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☞ I think my ſelf oblig'd in Juſtice to Recommend Mr. *John Rowley*, Mathematical Inſtrument maker under St. *Dunſtan*'s Church in *Fleet-Street*, to all Lovers of Art and Judges of Good Work, as the Beſt Workman I know, and of whom all ſorts of Mathematical Inſtruments may be had at Reaſonable Rates.

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John Harris.

Short, but yet Plain

ELEMENTS
OF
GEOMETRY.

SHEWING

How by a Brief and Eaſie Method, moſt of what is Neceſſary and Uſeful in *Euclide*, *Archimedes*, *Apollonius*, and other Excellent *Geometricians*, both Ancient and Modern; may be Underſtood.

Writen in *French* by *F. Ignat. Gaſton Pardies*

And rendred into *Engliſh*.
By JOHN HARRIS M. A. and F. R. S.

The *Third Edition*, in which are again many New Propoſitions, Additions and uſeful Improvements: The Problems being now placed every where in their proper Order, and the whole Accommodated to the Capacities of young Beginners.

London, Printed by *E. Midwinter*, for *R. Knaplock* at the *Angel*, and *D. Midwinter* at the *Roſe* and *Crown* in St. *Paul's Church-Yard*. 1705.

BRITISH MUSEUM

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TO

My Worthy Friend

Charles Cox Eſquire,

Member of Parliament for the Burgh of *Southwark*.

Dear SIR,

AMONG the many Obligations You have conferr'd upon me, I account it not the leaſt, that You have given me a Riſe *to revive my* Mathematical Studies; *in which as I have formerly ſpent ſome time, ſo I know of no more* Uſeful way *of employing my leiſure Hours.*

And indeed, Sir, the Diverſion *and* Advantage *I have lately reaped from them hath (by the Divine Bleſſing) both ſupported me under, and in a good Meaſure carried me through ſuch Preſſures and Difficulties as I once almoſt deſpaired of Surmounting.*

The Mathematick Lecture *which you at firſt ſet up* Gratis *in your* Burgh, *and which out of an uncommon* Generoſity, *you did afterwards* * *remove into the City of* London, *is a Demonſtrative Proof both of your ſincere Endeavour to promote the good of your Country, and alſo of your Capacity to do it the beſt way. And as I have already in a good degree, ſo I hope to ſee ſuch Effects from ſo Noble a Deſign, as will render your Name juſtly Honourable to Poſterity as well as to this preſent Age, Sir you know* your ſelf *and* Me *too well, to take this for Flattery. 'Tis what* Truth, Juſtice *and* Gratitude *Oblige me to ſay.*

I ſhall only add, that I am again glad of this opportunity to ſhew the Juſt Eſteem I have of your Merit, and the equal Regard I have for your Friendſhip. I am,

SIR,

Your Moſt Obliged

Humble Servant

John Harris.

THE TRANSLATOR TO THE READER.

AFter frequent Peruſal, and mature Deliberation on this Book; I judge it to
the plaineſt, ſhorteſt, and yet eaſieſt Geome-
y I have ever ſeen Publiſhed: And there-
e I thought it very well worth my while to let
appear a third time in our own Language, as
had already done twice before in the Latin
ongue. And 'tis ſo well eſteemed of, by very
mpetent Judges amongſt Us, as to be read in
r Univerſities, by Tutors to their Pupils:
d alſo, which is not uſual with Books of
is kind, there hath been two entire Impreſſi-
s ſold off in a little more than as many Years
ime.

As to the Tranſlation; I have by no means
liged my ſelf ſervilely to follow the French
y of Expreſſion; for indeed a Literal Ver-
n of a Book out of any Language will be ſcarce

Intelligible in Engliſh. *I have therefore all along aimed rather to give you* F. Pardies *Senſe, than his Words; and have made him ſpeak what I judge he would have done, had he wrote in our Language. I have made no ſcruple to add any thing that I ſaw neceſſary to render him clear and intelligible; and particularly in this* Third Edition *I have added many things, which were not in the two Former.*

As the ſecond Book of Euclide *about the Power of Lines; The Menſuration of the Surfaces, of Solids,* Archimedes *his Proportion of the Inſcribed Cone and Sphere to the Circumſcribing Cylinder; the Figures of the* 5 *Regular Bodies; ſeveral Additions and improvements in the Doctrine of Proportion: The Menſuration of the* Fruſtrums *of Pyramids and Cones; ſome new Properties of a Right-angled Triangle, and of the Circle,* &c. *I have alſo, in this Edition, left out ſome more of* Pardie's *Propoſitions which on repeated experience in Teaching, I have found leſs uſeful; as alſo all the Elements of plain Trigonometry, which I had before added to his Ninth Book; becauſe I have publiſhed a ſmall Treatiſe on that Subject by it ſelf; and my chief aim* now *hath been to lead the Learner into a little more Abſtracted and Conciſe, tho' a moſt* *uſeful and Univerſal Method of Demonſtration; introducing now and then a little of* Algebra, *that I may thereby engage the Reader in a Love of, and Value for that moſt Noble and Wonderful Science: and to give him a good Foundation to build upon; and a ſufficient Riſe thereby to carry him into Fluxions and the new Methods of Inveſtigation and Demonſtration, where he will find ſufficient Satisfaction. Nor need he be diſcouraged at the attempt, for 'tis well known that I have taught ſeveral Perſons to underſtand the Elementary Parts of all Mathematicks ſo well, that they have been, able to go on every where without the Aſſiſtance of any Maſter, in leſs than a Years Time.*

Pardie's *Advice to thoſe who would Underſtand Geometry.*

1. THey ought to enure themſelves to conſider well the *Figures*, at the ſame time as they Read the *Propoſitions*. There will be ſome Labour and difficulty at firſt, but they will break thro' it in two or three Days.

2. They ought not to be diſcouraged, if they meet with ſome things which they do not underſtand at firſt; Geometry is not ſo eaſie to be attained, as Hiſtory.

3. If, after they have Read and Conſidered attentively any Propoſition, they find they don't underſtand it; let it be paſſed over, it will probably be Intelligible by reading farther, or at leaſt when they have gone over the whole, and have began to Read it over anew. There are indeed many things in Geometry, that will never be well underſtood at firſt Reading over.

4. The Numbers which are within the Parentheſis, *v. gr.* (3. 14). ſhew that the Matter there ſpoken of hath been proved elſewhere,

viz. in this Inſtance, in the fourteenth Article of the *Third* Book; And they ought always to mind the Number of the Article, and to conſult the Places referred to, that ſo they may gain the Demonſtration of what they Read.

5. When they meet with any Words which they don't underſtand, they may conſult the Table at the end of the Book.

6. 'Tis good to have a Maſter at firſt, to Explain to them the Nature and Manner of the Demonſtrations: For by that means they will Underſtand the thing both much eaſier and much ſooner, than they can do by Reading by themſelves.

A

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N. B. The firſt Figure ſhews the Book, the Second the Article.

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The End

The Characters, Marks, Signs or Symbols here uſed are only theſe,

= EQual to.
+ More or adding.
— Leſs or Subſtracting.
∷ The Mark of four Quantities being diſcretely Proportional Geometrically.
∻ The Mark for Continual Proportion, or Geometrick Progreſſion.
∶∶∶ The Mark for Arithmetical Proportion.
× The Mark for Multiplication.
□ Square.
▭ Rectangle.
△ Triangle.
∠ Angle.
‖ Parallel.

ELEMENTS OF GEOMETRY.

BOOK I.

Of Lines and Angles.

1. BY the Word *Quantity*, which in the general is the Subject of *Geometry*, we mean that whereby one Thing, being compared with another of the ſame Nature, may be ſaid to be Greater or Leſs than, Equal or Unequal to it: As Extenſion, *i. e.* Length, Breadth or Thickneſs, Number, Weight, Time, Motion, And all thoſe things, which are capable of being ſo compared as to More or Leſs, are the Object of Geometry.

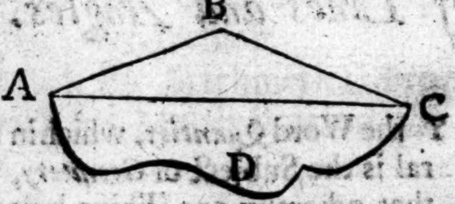
2. We deſign nevertheleſs to conſider now only Extenſion; as being that which ſerves for an *Example* and *Rule* to Meaſure all other Quantites by.

3. That Quantity which, being ſuppoſed without any Breadth or Thickneſs, is extended only in Length, is called a *Line.* That which hath both Length and Breadth, (but is ſuppoſed to have no Thickneſs) is called a *Surface* or *Superficies:* And that which hath Length, Breadth and Thickneſs, is called a *Body* or *Solid.*

4. A *Point* is that, which is conſidered as having no manner of Dimenſions; and as being Indiviſible in every reſpect. The Ends or Extremities of Lines, as alſo the middle of them are Points.

5. There are *Strait Lines*, and there are *Crooked* or *Curved* Ones: Alſo there are Even and Plain Surfaces which are called *Plan's*; and there are *Crooked* or *Curved* Ones: Which like a Vault, (*or the Tilt of a Boat or Waggon.*) are *Convex* above, and *Concave* below.

The Generation of Lines may be eaſily conceived to be made by the Motion or Fluxion of a Point, as A.



Which if it move directly from the Term A, to the Term C, or go the neareſt or ſhorteſt way poſſible, it then forms what Geometers call a *Right* or *Strait Line.*

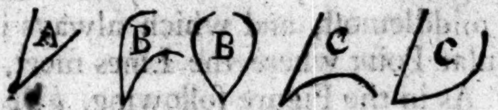
If it go firſt directly to B, and then alſo the neareſt way to C; it forms two right Lines, A B and B C, which taken together, are longer than the Line A C; and conſequently, two ſides of any Triangle muſt be longer than the Third.

If the Point A, move not in one or more right or ſtrait Lines towards C, it muſt go Crooked, and ſo will form a Curve or crooked Line, as A D C.

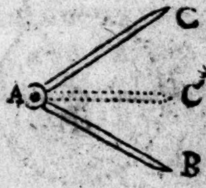
And from hence alſo 'tis plain, that any two Points moving with equal Velocity, will in the ſame Time generate equal Lines.

6. When two Lines meet in a Point, The *Aperture, Diſtance* or *Inclination* between them is call'd an *Angle.* Which when the Lines forming it are right or ſtrait Ones, is called a *Rectilineal Angle*; as A. But they are crooked, 'tis called a *Curvilineal One*; as B. And when one is ſtrait and the other crooked, 'tis called a *Mixt Angle*; as C.

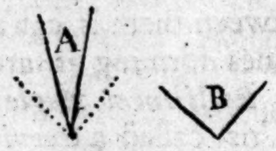
N. B. *The Lines forming any Angle, are called its Legs.*



7. That Angle is ſaid to be leſs than another, whoſe Legs are more inclined to, (*or nearer to*) each other. Let there be two Lines A B and A C meeting in the Point A. If you imagine thoſe Lines to be moveable like the Legs of a pair of Compaſſes, and yet faſtned together in A, as with a Joint; 'Tis eaſie then to conceive, that the further they are opened or parted from one another, the greater will be the Angle between them: As on the Contrary, the nearer they are brought together, the more will they incline towards each other, and ſo the Angle between them muſt be the leſs.

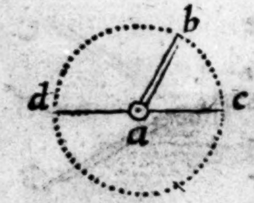


8. It muſt therefore be obſerved, that the Quantity of Angles, is by no means Meaſured by the Length of their Legs, but by their Inclination. Thus, *v. gr.* The Angle B is bigger than A; tho' the Legs of the Angle B, are much ſhorter than thoſe of A: But then thoſe of A are much more inclined to each other, than thoſe of B. And to apprehend this the better, imagine the Angle B to be put upon A, as you may conceive by the prickt Lines about A, which repreſent the Legs of B lying on it. For 'tis plain the Angle A will be eaſily contained within B; and that its Legs, are much more inclined to one another, than thoſe of B, and therefore it is leſs than B.



9. An Angle is uſually marked by three Letters, of which the middlemoſt, and which always is placed at the Angular Point where the Lines meet, denotes the Angle. As in the Figure following, *bac* denotes the Angle made by the two Lines *b a* and *c a* meeting in the Point *a*.

10. If we imagine the Line *a b* faſtned by its end, *a* in the middle of the Line *d c*, but yet ſo as to be moveable on *a*, as on a Center: If then you conceive it to be moved quite round, till it arrive at the place where it began, the Point *b* will deſcribe a Curve Line, which is uſually called a Circle; but 'tis rather the *Circumference of a Circle*; for properly ſpeaking, the Circle is the *Space* contain'd within the Circumference.



11. Any part of the Circumference is called an Ark, as *b c*.

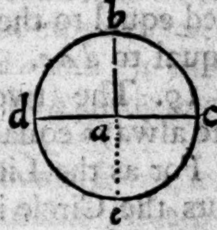
12. The Line *d c* (*paſſing through the Center*) and terminated by the Circumference, is called the *Diameter*.

meter, and divides the Circle into two equal parts. Alſo every right Line paſſing thro' the Center *a* (*and Terminated at each end by the Circumference*) divides the Circle into two equal Parts, and will be a Diameter.

13. The Line *a b* or *a c*, or any other drawn from the Center to the Circumference, is called the Radius or Semi-diameter.

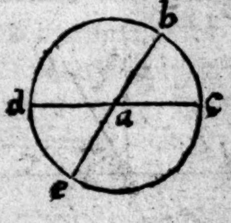
14. All *Radius*'s or *Semi-diameters* (*of the ſame or equal Circles*,) are equal. (*as is plain from the Geneſis of a Circle given in* Art. 10.)

15. When the end *b* of the Radius *a b* is equally Diſtant from the two ends of the Diameter *d c*; That is, when the Point *b* is in the very middle of the Semi-circumference *d b c*; then will *b a* make two Angles with *d c* that are called Right Ones: Which are equal to one another, that is, the Angle *d a b* is equal to *b a c*. And if the Line *b a* be produced below to *e*, it ſhall then (with *d c*) make four Right Angles; and it will be another Diameter; which with the former *d c* will divide the Circle into four equal Parts.

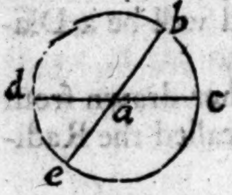


16. Then thoſe Lines are ſaid to be Perpendicular one to another, *viz.* *b a* to *d c*, and *d a* to *b e*.

17. But if *b* be nearer to one End of the Diameter (or Right Line) *d c*, than it is to the other, it is then ſaid to fall on the other Obliquely; and it makes with *d c* two Angles that are Unequal: Of which the Leſſer *b a c* is called Acute, and the Greater *d a b* is called Obtuſe.



If the Line *a b* be produced to *e*, it will be a new Diameter, and will make below two other Angles: So that in the whole there will be four Angles; of which thoſe two that touch only in the Angular Point, as *b a c* and *e a d*; as alſo, *d a b* and *e a c*, are called Vertical or Oppoſite Angles. But thoſe that have one Leg common to both, as *d a b* and *b a c*; and *b a c* and *e a c*, are called *Adjoining or Contiguous Angles.*

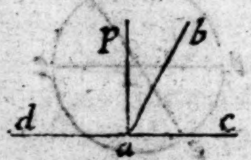


18. Thoſe Angles which (at equal Diſtances from the Angular Point) are ſubtended by equal Arks, are alſo equal themſelves. As if the Ark *b c* be proved equal to the Ark *d e*, then will the Angle *b a c* be equal to *d a e*.

19. The two *Contiguous Angles* taken together, are always equal to two Right Ones.

For as the Line *d c* is a Diameter, and therefore cuts the Circle into two equal Parts, the two Arks *d b* and *b c* taken together, will be equal to a Semicircle. Wherefore the two Angles *d a b* and *b a c* together will be equal to two Right Ones, becauſe they compleat the whole Semi-circle, as two Right Ones do. (*Art.* 15.)

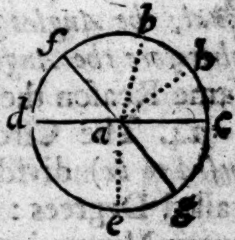
20. So that this Propoſition is of Univerſal Truth, *That one Right Line falling on another, makes the Contiguous Angles* (together) *equal to two Right Ones.* For if the Lines are *Perpendicular* to each other, as *p a* is to *d c*. Then 'tis plain the Angles muſt be Right (by the 15.) And if the Line fall Obliquely, as *b a* doth, then indeed the Angles are Unequal: But as much as the Obtuſe one *d a b* exceeds one Right Angle, by ſo much is the Acute one *b a c* exceeded by the



the other Right One. So that the Smalneſs of one is compenſated by the Greatneſs of the other.

21. Hence alſo, it follows Converſely, (for where-ever the Property is found, there the Thing is, in Geometry) that if two Angles which have one Leg Common to both, do make Angles equal to two Right Ones, their other Legs do make but one Right Line. Let the Angles *d a b* and *b a c* be *(together)* equal to two Right Ones. Then I ſay, that the Lines *d a* and *a c* do join ſo together, as to make one Right Line (*vid. Fig. in Art.* 17.) which is clear from what hath been ſaid. For if on the Center *a* you deſcribe a Circle *d b c e*, the two Arks *d b* and *b c* will be equal to a Semi-circle, becauſe the two Angles *d a b* and *b a c* are ſuppoſed to be equal to two Right Ones. Wherefore the Lines *d a* and *c a* will make a Diameter, and conſequently be joined into one Right Line.

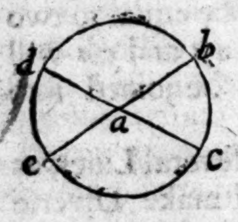
22. If from the Point *a* you draw ſeveral Lines, as *a d*, *a f*, *a b*, *a h*, *a g*, &c. they will make Diverſe Angles; and all thoſe Angles taken together, be they more or leſs, will be equal to four Right Ones. For 'tis clear all theſe Angles together do compleat the Circle *d b c c*, whoſe Circumference they divide into as many Arks as there are Angles. Now all theſe together are equal to four Quarters of a Circle; which is as much as to ſay, that all the Angles are equal in the whole to four Right Ones; for ſo many Right Angles do alſo compleat the Circle.



AXIOM I.

If to, or from equal Things, you add or Subtract Equals, the Sums or Remainders will be Equal.

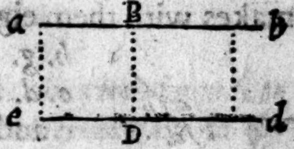
23. The Vertical or Oppoſite Angles are equal. Let there be two Right Lines *d a c* and *b a e* (*croſſing or cutting one another in the Point* a,) I ſay, the Angle *d a e* is equal to *b a c*. For the Ark *d b* with the Ark *b c* makes a Semi-circle; and ſo doth the ſame Ark *b d* with the Ark *d e*. Therefore the Ark *b c* muſt be equal to *d e*; becauſe the Ark *d b* continues the ſame, whether it help to compleat the Semicircle with *d e* or *b c*: (*wherefore being taken away from both, it muſt leave the Ark* d e *equal to* b c. *But if the Arks be equal, the Angles ſubtended by them muſt be ſo too, and therefore the Angle* d a e *is equal to* b a c.) And by the ſame Reaſon the Angle *d a b* will be equal to *e a c*.



24. The Circumference of every Circle is (*ſuppoſed to be*) divided into 360 equal Parts, which are called Degrees: And every Degree into 60 Minutes, every Minute into 60 Seconds, every Second into 60 Thirds, and ſo on infinitely. And to determine the Quantity of every Angle, we compute the Degrees that (*the Ark, which is its Meaſure*) doth contain, *v. gr.* When we ſpeak of an Angle of 90 Deg. we mean a Right Angle; becauſe the Right Angle contains the fourth Part of the whole Circumference, which is 90 Deg. the fourth Part of 360. So an Angle of 60 Deg. is an Angle that contains two Thirds of a Right One.

25. (*Degrees are marked either with* Degr. *or uſually with a ſmall Cypher over the laſt Figure, as* 60°.) Minutes with a ſmall Line, as 50′, Seconds with two ſuch, as 30″, Thirds with three ſuch, as 25‴, *&c.* So that 25° 32′ 43″, is to be read, 25 Degrees, 32 Minutes, 43 Seconds.

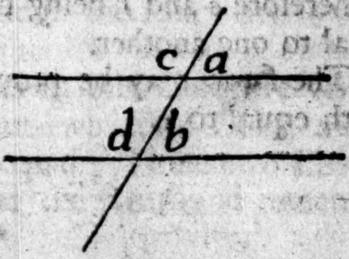
26. Two Lines are ſaid to be Parallel, when they run always equi-diſtant from each other. Thus the two Lines *a b* and *e d* are Parallel, if they are equally diſtant from each other in *a e*, in BD, in *b d*, and in all other Places.



27. This Diſtance is always Meaſured by a Perpendicular; as if from the Point *a* you imagine the Line *a e* to fall Perpendicular to *e d*; as alſo doth the Line *b d* on the ſame Line; we naturally conceive that if thoſe two Perpendiculars are of the ſame Length, or equal; the two Lines *a b* and *e d* are equally diſtant from each other in thoſe two places, which is ſelf-evident and needs no Proof.

28. Two Parallel Lines being continued infinitely, yet can never meet. For being always equally diſtant, there may any where be drawn between them a Perpendicular equal to *a e* or *b d*, and conſequently, they can never meet.

29. Two Parallel Lines, have the ſame Inclination, one as the other, to any right Line that croſſes them both.

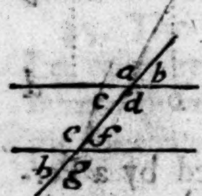


That is, the Angle *a* will always be equal to *b*, and *c* to *d*; for the Interſecting Line being ſuppoſed inflexible, as is the Caſe of all Mathematick Lines, it cannot bend to, or from one

Parallel more than it doth to or from the other: And neither of theſe Lines can alter its Poſition in reſpect of the Croſſing Line, for then the Paralleliſm would be deſtroyed, which Contradicts the Suppoſition.

And this is the firſt Property of Parallel Lines.

30. Whenever a Right Line cuts two Parallels, it makes with them eight Angles: Of which, four *a. b. h. g.* are *External*; and the other four *c. d. e. f.* are *Internal.* The Angles *c* and *f*, as alſo *d* and *e*, are called *Alternate.* The Angles *e* and *a*, as alſo *f* and *b* are called the Internal and *Oppoſite on the ſame ſide.* And the Angles *d* and *f*, as alſo *c* and *e*, are called the *Internal Angles on the ſame ſide.*



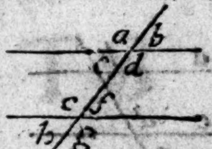
AXIOM II.

Things equal to a Third, are equal to one another.

31. The Alternate Angles *c* and *f* muſt be equal; as alſo *e* and *d*, for *c* is equal to the Vertical Angle *b*, and *b* is equal to the Internal one *f*, by the laſt *Prop.* Wherefore *c* and *f* being both equal to *b*, muſt be equal to one another.

The ſame may be proved of *e* and *d*, which are both equal to *a*.

32. When a Line falls on two Parallel Ones, it makes the Internal Angles on the ſame ſide equal to two Right Ones.



I ſay, the Angle *d* with *f*, is equal to two Right Ones: Becauſe *f* is equal to *c* (by 31) and *c* and *d* together are equal to two Right Ones (by 20.) Therefore *f* and *d* together muſt be equal to two Right Ones, which was to be prov'd.

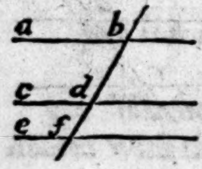
(The ſame way may c *and* e *together be proved equal to two Right Ones; for* c *and* d *taken together are ſo (by* 20*) but* d *is equal to* e *(*31.*) Therefore* c *and* e *are equal to two Right Ones.)*

33. One Propoſition, is called the Converſe of another; when after a Concluſion is drawn from ſomething Suppoſed, in the Converſe Propoſition that Concluſion is Suppoſed; and then that which was in the other Suppoſed, is now drawn as a Concluſion from it. For Example: We ſay here, if two Lines are Parallel, *(and another Croſs them,)* the Angles *d* and *f* together, are equal to two Right Ones: Where we ſuppoſe the Lines to be Parallel, and from thence conclude thoſe Angles muſt be equal to two Right Ones: But the Converſe is thus, If the *Internal Angles on the ſame ſide*, *d* and *f* together, are equal to two Right Ones; Then thoſe Lines are Parallel: Where after we have ſuppoſed the Angles equal to two Right Ones, we conclude the Lines are to be Parallel.

34. Converſe Propoſitions in this Caſe are very True; as that, if a Line cut two other Lines, and makes the Alternate Angles equal; Thoſe two Lines are Parallel: Which I deſire the Reader to remember.

35. If two Lines are Parallel to a third Line, they are ſo to one another.

Let the Line *a b* be Parallel to *c d*; and let *e f* alſo be Parallel to the ſame Line *c d*; I ſay, *a b* is Parallel to *e f*: For if you draw a Line as *b d f* cutting them all Three; the Angle *b* will be equal to *d* (by 31.) and the ſame will be equal to *f* (by 31.) becauſe *e f* is alſo Parallel to *c d*. Wherefore the Angle *b* muſt be equal to *f*: Becauſe by Axiom 2, *if two things are equal to a third, they are ſo to one another*. But if the Angle *b* be = *f*, then the Line *a b* is Parallel to *e f* (by 34.)



BOOK

ELEMENTS OF GEOMETRY.

BOOK II.

Of Triangles.

1. A *Figure* is a Space, compaſſed round on all Sides. And if the Lines which Terminate it, are all Right Ones, 'tis called a Rectilineal (or Right Lined) Figure: If they are Crooked, 'tis called a Curvilineal; and if they are partly Right Lines, and partly Crooked, 'tis called a Mixt Figure.

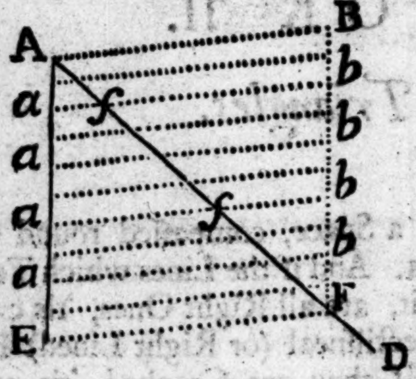
2. There are Plane Figures, which are Plane Surfaces, and there are Solid Ones, which have three Dimenſions. But we ſpeak here only of Plane Surfaces, or Plane Figures.

3. All the Lines which encompaſs any Figure taken together, make that which is called the Circumference, Perimeter, or the Compaſs of the Figure.

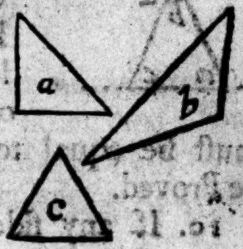
4. Of all Curvilineal or Mixt Plane Figures, in Common Geometry, we conſider properly only the Circle, or a part of a Circle terminated on one ſide by an Ark, and on the other by one or more Right Lines.

5. Of Rectilineal Figures, the moſt Simple are Triangles, which are Terminated by three Right Lines (*and no more*) making as many Angles.

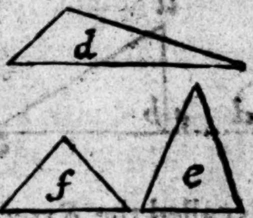
If a Right Line, (A B) having one of its Ends or Points (as A) in the Vertex or Top of the Angle, E A D be moved downwards, with a Motion always Parallel to its ſelf, ſo that the Point A ſhall always keep in or touch the Line A E, until it come to be all of it within the Legs of the Angle E A D; that is, till it come to be in the Situation E F; that Line ſhall in its Motion continually cut the Line A D, and at length deſcribe the Triangle E A F within the Legs of the Angle; as alſo another equal to it (A D B) on the other ſide of the Line A D. The Parts of which latter Triangle ſhall continually Decreaſe, as thoſe of the former A E F, do continually Increaſe. And the Line A B ſhall alſo deſcribe with its whole length the Quadrilateral Figure A E F B; which will be divided into two equal Parts by the Diagonal Line A F.



N. B. The Line A B may be called the *Deſcribent*, and A E the *Dirigent*, becauſe the latter directs the Motion of the former.



6. A Triangle as *a*, which hath one Right Angle, is a *Right-angled* Triangle; if it have one Angle Obtuſe, 'tis called an *Obtuſe-angled* One, as *b*; and if all its three Angles are Acute, 'tis called an *Acute-Angled* Triangle, as *c*.

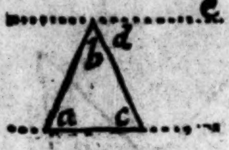


7. If a Triangle have all its three ſides Unequal, 'tis called a *Scalene* as *d*. If it hath two ſides equal, 'tis called an *Iſoſceles* as *e*; And if all the three ſides are equal, 'tis called an *Equilateral* one, as *f*.

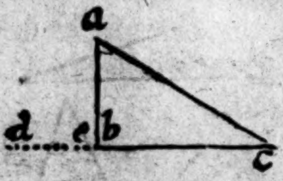
8. When two ſides of a Triangle are conſidered, they may be called its *Legs*, and the third ſide may then be called the *Baſe*. But any one ſide may be called the *Baſe*, (*tho' we uſually and moſt properly call that ſo, which lies Parallel to the Horizon, and which is next to us.*)

9. In every Triangle, the three Angles taken together, are equal to two Right ones.

Let the Triangle be *a b c*: I ſay, that the Angle *a* added to the Angle *c*, added to the Angle *b* (*or the Sum of all three*) are equal to two Right ones. For let *d e* be drawn Parallel to the *Baſe a c*, then will thoſe two Parallel Lines be cut by the Line *b c*; and conſequently the Alternate Angles *c* and *d* will be equal to each other (by 1. 31.) Moreover the Line *b a* falling on, or cutting the ſame Parallels *d e* and *a c*, will make the two Internal Angles on the ſame ſide equal to two Right ones ; that is, *a* added to *a b e* are equal to two Right Angles (by 1, 32.) But the Angle *a b e* contains the two Angles *b* and *d*. So that the Angle *a* added to *b* added to *d* will be equal to two Right ones. But *c* being equal to *d*, it will follow that *a* added to *b* added to *c*, or the Sum of all three together, muſt be equal to two Right ones ; Which was to be Proved.



10. If any ſide of a Triangle be produced, or drawn out, the external Angle will be equal to the two Internal Oppoſite Angles, (*taken together.*) Let the Triangle be *a b c*, whoſe Baſe *c b* draw out to *d*, by which means a New Angle as *e* will be made, which is called the *External* Angle of that Triangle. Then I ſay, that That External Angle *e*, is equal to both the Internal and Oppoſite ones *a* and *c*.



For thoſe Angles *a* and *c* together with *b* are equal to two Right ones (by the Precedent,) and ſo alſo are *e* and *b* by (1. 20.) wherefore *e* muſt be equal to *a* added to *c*, becauſe together with *b*, it makes two Right Angles, as they do. Q. E. D.

COROLLARIES.

1. The Sum of the three Angles of all Triangles is the ſame.

2. No Triangle can have above one Right or Obtuſe Angle.

3. If in any Triangle one Angle be Right, the other two muſt be Acute.

4. If in any Triangle there be one Angle equal to both the others, that muſt be a Right One.

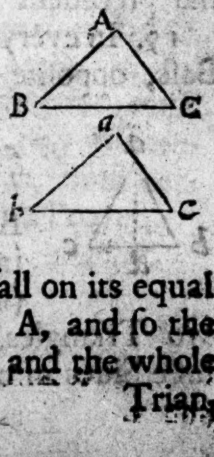
5. If you know the Degrees of one Angle in any Triangle, you know the Sum of the other two; for 'tis what is wanting of 180°, and if the Sum of any two be known, the Quantity of the Remainder is known.

6. Hence if two Triangles have any two Angles reſpectively equal to one another, the remaining Angles muſt alſo be equal.

7. The Angle of an Equilateral Triangle is $\frac{1}{3}$ of two Right Angles, or $\frac{2}{3}$ of one Right Angle, equal to 60°.

8. Hence 'tis very eaſie to Triſect a Right Angle, by making on one of the Legs an Equilateral Triangle.

11. If a Triangle A B C hath two Sides, A B and A C, equal to two others *a b* and *a c* in another Triangle, and if alſo the Angle A be equal to *a*, I ſay, the third Side B C ſhall be equal to *b c*; the Angle B equal to *b*, the Angle C to *c*, and the whole Triangle A B C to *a b c*.



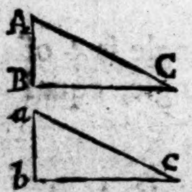
For if we imagine the Triangle *a b c* to be placed upon A B C, ſo that the ſide *a b* ſhall lie exactly on its equal A B ∴ Then muſt the ſide *a c* fall on its equal A C, becauſe the Angle *a* is equal to A, and ſo the Point *c* will fall on C, and *b* upon B, and the whole

Triangle *a b c* on the Triangle A B C; becauſe all things ſo exactly anſwer, that nothing of the Upper Triangle can fall beſides the Under one.

12. Figures which do thus meet, fit, or anſwer to each other exactly, when they are placed one upon the other are called Congruous Figures, *Quia mutuo ſibi Congruunt.*

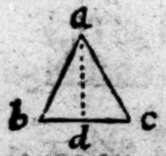
And therefore the third Axiom is, *Quæ ſibi mutuo Congruunt ſunt Æqualia*; *i. e.* Thoſe Figures which placed one upon another, do anſwer to, and cover one another exactly are equal.

13. It is alſo true, That, if a Triangle hath all its three Sides equal to the three Sides of another Triangle, all the Angles alſo in one, ſhall be equal to thoſe in the other: And all the Space which one Triangle contains, ſhall be equal to that contained in the other: As if A B be equal to *a b*, A C to *a c*, and B C to *b c*: I ſay, that the Angle A ſhall be equal to *a*, B to *b*, and C to *c*; and the whole Triangle A B C, to *a b c*; this needs no other Proof.



14. If the Angle A be equal to *a*, the Angle B to *b*, and the Side A B to *a b*: Then ſhall the Side A C be always equal to *a c*, B C to *b c*; and the whole Triangle A B C to *a b c*: Which is eaſie to Prove by the precedent Propoſitions.

15. In every *Iſoſceles* Triangle, the Angles at the Baſe, oppoſite to the equal Legs, are equal.

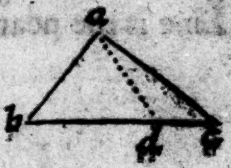


Let the Triangle be *a b c*, whoſe Legs *a b* and *a c* are equal: I ſay, the Angle *b* is alſo equal to *c*. For imagine the Baſe *b c* divided into two equal Parts in the Point *d*, then will the Line *a d* (*which let be drawn*) make of the whole, two Triangles, *a b d* and *d a c*, which will have all three Sides in one, equal to thoſe in the other: For *ab* is equal to *ac* by the Suppoſition, and *bd* is equal to *dc*, and *ad* is common to both. Wherefore (by 2. 13.) the whole Triangle *bad* is equal to *dac*, and the Angle *b* is equal to *c*; which was to be proved.

16. In an *Iſoſceles* Triangle, if a Line drawn from the Angle at the top do (*Biſſect or*) divide the Baſe into two equal Parts, it is both Perpendicular to the Baſe, and alſo Biſſects the Angle at the top into two equal Parts. For (*vid. Fig. precedent.*) the Angle *adc* is equal to the Angle *adb* (by the laſt) and conſequently they muſt be both Right Ones; and therefore the Line *ad* is perpendicular to the Baſe *bc* (1. 16) and the Angle *dac* will be equal to *dab* (by the laſt Prop.)

17. In every Triangle the Greater ſide is always oppoſite to, or ſubtends the Greater Angle.

In the Triangle *abc*, let the ſide *bc* be longer than *ba*; then I ſay, the Angle *bac* ſubtended by the Greater Side *bc* is bigger than the Angle *c*, which is ſubtended by the leſſer Side. For let *bd* be taken equal to *ba*, then will *abd* be an *Iſoſceles* Triangle; whoſe Angle *bad* will be equal to *bda* (2. 15.) But the Angle *cab* is bigger than *bad*; (*The whole being greater than the Part*) and therefore muſt be bigger than *bda* (*which is equal to* bad). Now the Angle *adb* is an External Angle in reſpect of the little Triangle *adc*; and therefore muſt be bigger than the Internal one *c* (by 2. 10.) Wherefore the Angle *bac* being bigger than *d*, muſt certainly be bigger than *c*; which was to be Proved.



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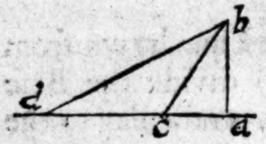
18. Of all Lines that can be drawn from a Point given to a Line given, the Shorteſt is the Perpendicular; and they are all Longer, according as they are farther diſtant from it. Let the given Line be *a d*, and the Point given *b*; let *b a* be perpendicular to *d a*; let alſo *b c* and *b d* be drawn. I ſay, that *b a* is the ſhorteſt Line that can poſſibly be drawn from *b*; and (for inſtance) is ſhorter than *b c* (*or any other that can be aſſigned*): and I ſay alſo, that *b d* is longer than *b c*.



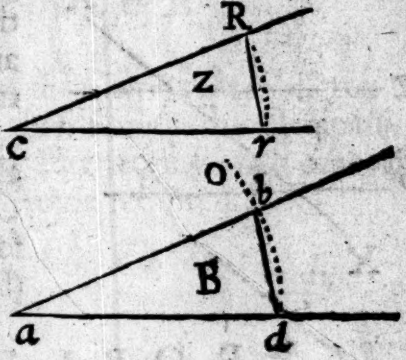
For in the Triangle *b a c*, the Angle *a* is a Right One, and conſequently, bigger than either of the other; becauſe they muſt neceſſarily be both Acute. (by *Cor.* 3. of *Art.* 10.) Therefore the Side *b c* is longer than *b a* (2. 17.) as Subtending a greater Angle.

So alſo in the Triangle *d b c*, the Angle *d c b* is Obtuſe, becauſe the Angle *b c a* is Acute: And conſequently the Side *d b* muſt be longer than *c b*, as Subtending a greater Angle (2. 17.)

19. In every Triangle any two Sides taken together are longer than the Third; becauſe a Right Line is the neareſt Diſtance between any two Points.

PROBLEM I.

On a Line given ad, *to make an Angle* B, *equal to a given one* Z.



Place the Compaſſes in *c* the Vertex of the given Angle, and deſcribe the Ark R*r*; then keeping them at the ſame diſtance, ſet one Foot in *a*, one end of the given Line, and with the other deſcribe the Ark *o b d*; ſet R*r* from *d* to *b*; And draw *a b*, ſo ſhall the Angle *b a d* or B be equal to Z.

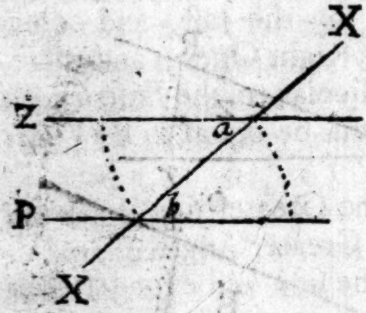
For the Legs of each are *Radii* of equal Circles, and the Line *b d* was taken equal to R*r*, wherefore the whole Triangles *c*R*r* and *a b d* muſt be equal (by 13.) and conſequently the Angle *a* equal to *c*.

PROBLEM II.

Hence the Practice of making all ſorts of Triangles, Equilateral, Iſoſcelar; or with any given Angles or Sides, will eaſily appear.

PROBLEM III.

A Right Line, as P, *being given, to draw thro'* a, *a Point given, the Line* Z a, *Parallel to it.*



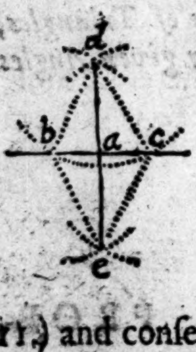
Through *a*, draw any Line, as XX, making any Angle, as *b*, with the given Line: Then make the Angle Z*a*X = to *b*, and Z*a* ſhall be the Parallel ſought.

For the Alternate Angles *a* and *b* are equal by Conſtruction; Wherefore Z*a* is Parallel to P*b* (by 1. 31.) Q. E. D.

PROBLEM IV.

To Biſſect or Divide a given Line c b *into two equal Parts in the Point* a.

Open the Compaſſes to more than $\frac{1}{2}$ the length of *b a c*, and with that diſtance make at each end of *b a c*, two pairs of interſecting Arks, as at *e* and *d*: Then drawing the Line *e d*, it will Biſſect the given Line in *a*.



For the Triangles *b e d* and *d e c* are equal (by 2. 13.) Wherefore the Angle *a d b* = *a d c*. Therefore the Triangles *a b d* and *a d c* will be equal alſo (by 2. 21.) and conſequently *a b* is = to *a c*. Q. E. D.

PROBLEM V.

By much the ſame Method may a Perpendicular, as a d, *be raiſed in the middle of any given Line, or one may be let fall from the Point* e or d, *to the given Line* a b c, And the Demonſtration is the ſame in all.

And after the ſame way of Practice may the given Angle b d c *be Biſſected.*

If ſetting one Foot of the Compaſſes in *d*, you take *d b* equal to *d c*: And then ſetting the Compaſſes in *b* and *c*, ſtrike the Arks Interſecting each other in *e*: So ſhall *d e* biſſect the Angle required.

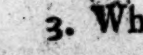
ELEMENTS OF GEOMETRY.

BOOK III

Of Quadrilateral Figures and Polygons.

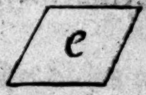
1. THOSE Figures, whoſe Sides are four Right Lines, and thoſe making four Angles, are called *Quadrilateral* or four-ſided Figures.

2. When the oppoſite Sides are Parallel, the Quadrilateral Figure is called a *Parallelogram* as *a*; but if not, 'tis called a *Trapezium* as B.



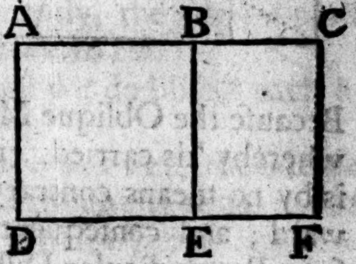
3. When the Parallelogram hath all its four Angles Right, 'tis called a *Rectangled Parallelogram*; or for brevity's ſake a *Rectangle* as *c*: And if the Angles are right, and the Sides are all equal, 'tis called a *Square*, as *d*.

4. If a Parallelogram hath all its Sides equal, but its Angles unequal, then 'tis called a *Rhombus*, as *e*.



5. If a Parallelogram hath neither its Angles nor ſides all equal, 'tis called a *Rhomboides*, as *a*.

The Generation of all Parallelogramick Figures will be eaſily conceived, if you ſuppoſe the *Deſcribent* A C, to be carried or moved along the Dirigent A D, in a Poſition always Parallel to it ſelf in its firſt Situation. For then, if the Angle A which the Deſcribent makes with the Dirigent, be a right One and A B be equal to A D: The Figure produced will be a Square. If A C be longer or ſhorter than A D, the Figure will be an Oblong, or a Rectangle.



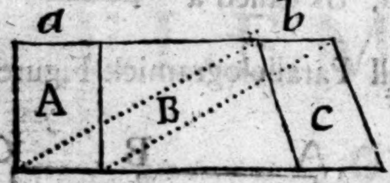
If the Angle at A be Oblique, only a Parallelogram at large will be Deſcribed: Which when the Deſcribent is equal to the Dirigent, the Figure will be a *Rhombus*, if unequal to it, a *Rhomboides*.

COROLLARIES.

I. Hence 'tis natural to ſuppoſe, that equal Lines moving thro' the ſame or equal Spaces will deſcribe equal Surfaces.

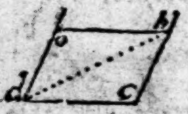
II. Equal Lines, with uniform or equable Motions (*i. e.* being neither Accelerated nor Retarded) in equal Times, will deſcribe equal Surfaces: And if they do thus deſcribe equal Surfaces, it muſt be in equal Times.

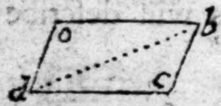
III. Hence alſo, if the Line *a* in a given Time deſcribe the Parallelogram A, and the equal Line *b* in the ſame Time deſcribe the Oblique Parallelogram B or C, whoſe Perpendicular Altitude is the ſame with that of A: Thoſe Parallelograms will be all three equal one to another. Becauſe the Oblique Motion, which the Line *b* hath, whereby 'tis carried, either to the right or left Hand, is by no means contrary to the direct Motion downward; and conſequently, the Line *b* will move the ſame Perpendicular Diſtance in the ſame time, with an equable Motion, whether the latter Motion be impreſſed upon it or not. Wherefore



IV. All Parallelogramick Figures, with equal Baſes and equal Perpendicular Altitudes, muſt be equal.

6. In every Parallelogram, the oppoſite Angles are equal. Let the Parallelogram be *o c*: I ſay, the Angle *o*, is equal to *c*; for the Angle *o* is equal to the Alternate one *b* (1. 31.) and the External one *b* is equal to the Internal one *c* (1. 31.) wherefore *o* is equal to *c*.





7. A Line as *d b* drawn acroſs the Figure from Angle to Angle, is called the Diagonal, and by ſome, the Diameter.

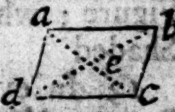
8. Every Parallelogram is divided into two equal Parts by the Diagonal. The Diagonal *b d* divides the Parallelogram *o c*, into the two equal Triangles *o b d* and *b c d*. For, 1. The Angle *o* is equal to *c* (3. 6.)

2. The Angle *o b d* is equal to *c d b* (1. 31.) and the ſide *b d* is common to both theſe Triangles, wherefore the Triangle *o b d* is equal to *c d b* (by 2. 14.)

9. In every Parallelogram, the oppoſite Sides are always equal.

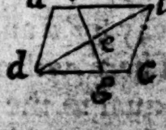
For, (*drawing the Diagonial* d b) the whole Triangle *d o b* will be equal to the Triangle *b c d*, by the foregoing *Prop.* And conſequently, the ſide *c d* muſt be equal to *o b*, and the Side *o d* to *c b*.

10. Two Diagonals *a c* and *b d* do biſſect each other in the middle at *e*.



For in the two Triangles *a e d* and *b e c*, the ſide *a d* is equal to *b c* (3. 9.) The Angle *e a d* is equal to *e c b* (1. 31.) and moreover the (*Vertical*) Angles *a e d* and *c e b* are equal alſo (1. 23.) *Wherefore the whole Triangle* a e d *is Reſpectively equal to the Triangle* b e c (2. 14.) And conſequently, the ſide *d e* is equal to *e b*, and the ſide *a e* to the ſide *c c*. The two Diagonals therefore biſſect each other in the middle. Q. E. D.

11. Every Right Line, as *f g*, paſſing through the middle of a Diagonal, divides the Parallelogram into two Equal Parts.

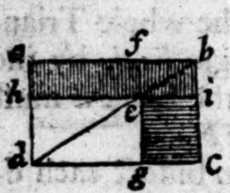


To demonſtrate which, the *Trapezium* or Irregular Quadrilateral Figure *f g d a* muſt be proved equal to the *Trapezium f g c b.* And that is thus done. 1. The Triangle *b e f* is equal to the Triangle *d e g*: For the ſide *d e* is equal to *e b* by the Suppoſition; and the Angle *e f b* is equal to *e g d* (1. 31.) and the oppoſite Angles at *e* are equal, wherefore the Triangle *e f b* is equal to *e d g* (2. 14). 2. The great Triangle *a b d* is equal to *b d c* (3. 8) wherefore if from the Triangle *a b d* you take away the little Triangle *f e b*, and inſtead of it put the Triangle *e d*

(*whic*

(*which is equal to* f e b) you will have the *Trapezium* *f a d g*, which will be equal to the Triangle *a d b*: That is, to juſt one half of the whole Parallelogram: (3. 8) which was to be proved.

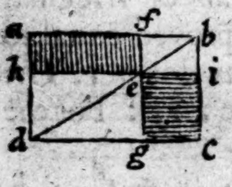
12. If in the Diagonal *d b* you take a point as *e*, and thro' it draw two Lines *h i* and *f g* Parallel to the two Sides of the Parallelogram, it will be divided by them into four leſſer *Parallelograms*, i. e. *f i*, *h g*: (which two are called the *Parallelograms about the Diameter*) and *a e*, *e c*: which other two are called the *Complements*. And thoſe two Complements with either of the Parallelograms about the Diameter make a Figure that is called a *Gnomon*. As you ſee in the Figure; where the *Gnomon* is diſtinguiſhed by being Shaded



13. In every Parallelogram the Complements are Equal. We muſt prove that *e a* is equal to *e c*.

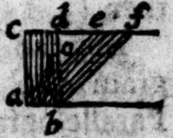
DEMONSTRATION.

The whole Triangle *a b d* is equal to the whole *b d c* (3. 8) And the little Triangle *e f b* is (for the ſame Reaſon) equal to *e b i*. And the Triangle *h e d* is alſo (by the ſame) equal to *e d g*. Wherefore if from the two equal Triangles *a b d* and *b d c* we take away equal things, *viz.* if from one we take away *e f b* and *d h e*, and from the other *e b i* and *e g d*, There will remain on one ſide the Parallelogram *e a*, equal to the Parallelogram *e c*, which remains on the other: which was to be proved.



14. Parallelograms having the ſame Baſe, and being between the ſame Parallels are equal.

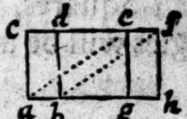
Let there be a Parallelogram *b c*, and another *a f*, both on the ſame Baſe *a b*; and let the Line *c d* when produced, be ſuppoſed to paſs by *e f*; So that the two Parallelograms ſhall be between the ſame Parallels, and Terminated by them; that is, between the two Parallels *c f* and *a b*. I ſay then that the Parallelogram *c b* is equal to *a f*.



For *c a* is equal to *b d*, and *a e* equal to *b f*, becauſe oppoſite ſides of Parallelograms, and the Angles at *c* and *d* equal (by 29. 1.) wherefore the Triangle *c a e* is equal to the Triangle *d b f*. Now if from each of theſe equal Triangles be taken the little Triangle *d o e*, and to the remainders be added the Triangle *a o b*, the Parallelogram *a d* will be equal to the Parallelogram *a f*. Q. E. D.

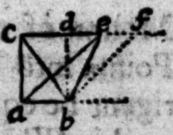
15. Parallelograms on equal Baſes *a b* and *g h*, and between the ſame Parallels *a h* and *c f*, are equal.

For if we imagine the third Parallelogram *f a* to be drawn; that ſhall be equal to the Parallelogram *a d*, becauſe on the *ſame Baſe a b* with it, and between the ſame Parallel Lines *a h* and *c f*. And that Parallelogram will alſo be equal to *e h*; becauſe it hath the ſame Baſe *e f* with it (it matters not whether you reckon the Baſe above or below) and it is between the ſame Parallels. Therefore *h e* and *b c* being both equal to the third Parallelogram *f a*, muſt be equal to each other.



16. *Triangles* on the ſame Baſe *a b*, and being between the ſame Parallels *c f* and *a b*, are always equal.

The Triangle *a b c* is equal to *a e b*: Becauſe if you imagine a Line *b d* drawn Parallel to *a c*, and another as *b f* drawn Parallel to *a e*; there will be made two Parallelograms

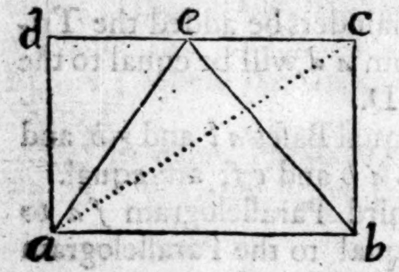


a c d b and *a e f b*; which being on the ſame Baſe and between the ſame Parallels, will be equal to one another (3. 14).

But the Triangle *a b c* is the half of the Parallelogram *a c d b*, and the Triangle *a b e* is the half of the Parallelogram *a e f b* (3. 8); wherefore, (*ſince the wholes are equal, the halves muſt*) and conſequently the Triangle *a c b* is equal to the Triangle *a e b*.

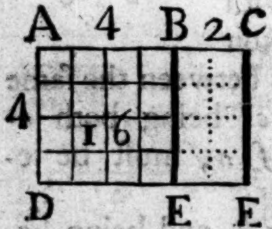
17. Triangles on *equal Baſes*, and between the ſame Parallels, are alſo equal; as is very eaſie to prove from (3. 15).

18. If a Triangle *a e b* have the ſame Baſe with a Parallelogram, and be alſo between the ſame Parallels, it ſhall be juſt the half of that Parallelogram. For it will ſtill be equal to *a b c* which is juſt half (3. 8) of *a b c d*.



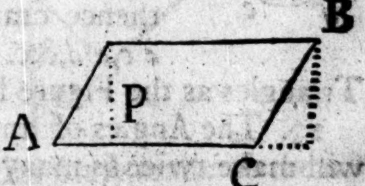
The Menſuration of all Squares, Rectangles, Parallelograms, and Triangles will be underſtood from what hath been delivered above. If you ſuppoſe

1. That the deſcribent A B or AC, before its motion, be divided into any determinate Number of equal Parts; and the Dirigent (now ſuppoſed to ſtand at Right Angles with it) into the ſame or any other Number of ſuch Parts; for then the Motion of the Deſcribent Line, thus markt out by Points into Unites will deſcribe a Square (if the Dirigent be equal to it,) and a Rectangle if it be unequal. Which Square or Rectangle will be divided into as many little Squares, as there are Unites in the

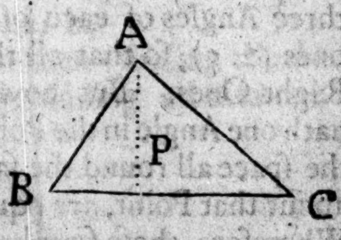


the Product of the Number of the Diviſions, or equal parts in one Line, multiplyed by thoſe in the other. That is A B 4 multiplyed by A D 4 produces 16; the Square of 4. And A C. 6. multiplyed by A D 4 produces 24; the Rectangle under A C and A D. So that what is a Product in Numbers or in Arithmetick; in Lines, or in Geometry, is called a Rectangle. And therefore you will find the Latine Writers of Geometry, when A C is to be multiplied by A D not ſaying *Multiplica*, but *Duc* A C in A D. That is, carry the Line A C along the Dirigent A D, in a Normal Poſition to it, till it come to end, and then it will Form the Rectangle A F = 24 where the Area of a Square is found, by multiplying the ſide A B into its ſelf.

The Area of any Rectangle as A F, is found by multiplying the Side A C by A D.



And ſince a Rectangle on the ſame Baſe and of the ſame Altitude with a Prallelogram is equal to it; to find the Area of any Parallelogram as A B you muſt multiply the Side A C by a perpendicular as P, let fall from the other Side to it.



And ſince every right-lined Triangle is the half of a Parallelogram or Rectangle of the ſame Baſe and Altitude: To find the Area of the Triangle A B C, you muſt multiply any Side as B C by a perpendicular as P, let fall to it from an oppoſite Angle, and take half the Product: or if either P or B happen to be even Numbers, multiply one by

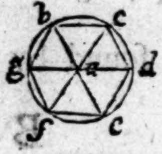
$\frac{1}{2}$ of the other, the Product is the Area of the Triangle.

19. A *Pentagon* is a Figure having five Sides and five Angles.

If all the Sides are equal, and conſequently the Angles, 'tis called a *Regular Pentagon.*

20. An *Hexagon* is a Figure of ſix Sides and Angles, an *Heptagon* of ſeven, an *Octagon* of eight, *&c.* which are all called *Regular* when they have equal Sides and Angles.

21. A *Polygon* in general ſignifies any Figure of many Sides and Angles; but no Figure is called by this Name, unleſs it have more than four, or five Sides.



22. Every *Polygon* may be divided into as many Triangles as it hath Sides, if any where within the *Polygon* you take a Point, as *a,* and from thence draw Lines to every Angle *a b, a c, a d,*&c. for they ſhall make as many Triangles as the Figure hath Sides.

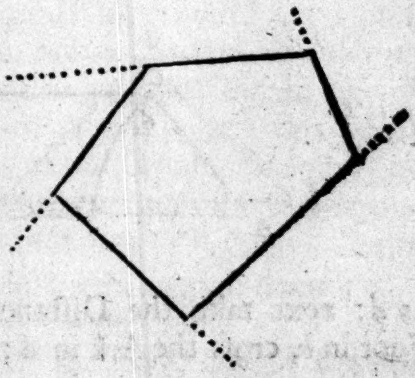
23. The Angles of any *Polygon* taken all together, will make twice as many Right ones, except four, as the Figure hath Sides. *v. gr.* If the *Polygon* have ſix Sides, the double of that is 12, from whence take four, there remains eight. I ſay that all the Angles of that *Polygon, viz. b, c, d, e, f, g,* taken together, are equal to eight Right Angles. For the Lines *a b, a c, a d,* &c. do divide the Figure into ſix *Triangles*; the three Angles of each of which are equal to two right ones (2. 9); ſo that all their Angles together make 12 Right Ones. But now, each of theſe ſix *Triangles* hath one Angle in the Point *a,* and by it they compleat the ſpace all round the ſaid Point. And all the Angles about that Point, are equal to four Right Ones (1. 22,) Wherefore thoſe four being taken from 12, (*The Sum of the Right Angles of all the ſix Triangles*) leaves eight, the Sum of the Right Angles of the *Hexagon,*

which make 8 times 90 or 720 Degrees; and therefore each Angle muſt be $\frac{1}{6}$ of that, *viz.* 120 Degrees.

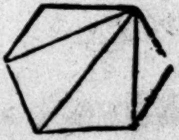
So that the Figure hath plainly twice as many right Angles as it hath Sides, except four; which was to be proved.

COROLLARY.

All the external Angles of any right-lined Figure, are equal to juſt four Right Ones: For drawing out the Sides as in the Figure, 'tis plain the Internal and External Angles together will make twice as many right Ones as the Figure hath Sides; but the Internal Angles are equal to all thoſe except four (by this prop.) Wherefore the External Angles muſt make up theſe four, and no more.



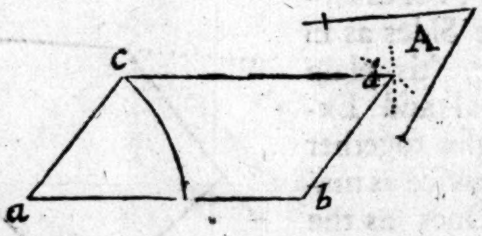
24. A *Polygon* may be divided alſo into *Triangles*, by drawing Lines from Angle to Angle. But then the Number of the Sides will exceed that of the *Triangles*. And hence the *Area* of any right-lined Figure may be found, by reducing it into *Triangles*, and then finding the Area of each *Triangle* ſeverally, adding all into one Summ.



PROBLEM I.

On a given Line a b, *to make a Parallelogram, having an Angle equal to a given Angle* A.

Make the Angle *c a b* = A. Then take *a b* in your Compaſſes, and ſetting one Foot in *c*, ſtrike an Ark



as *d*: next take the Diſtance *a c*, and placing one Foot in *b*, croſs the Ark in *d*: draw *c d* and *b d*, and it is done.

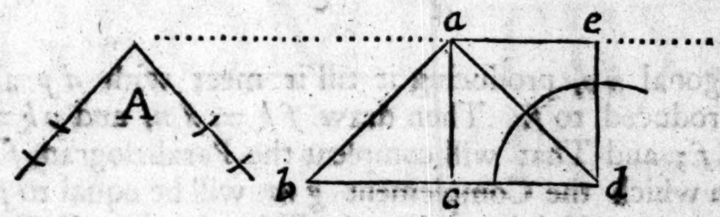
And thus alſo may the Line *c d* be drawn Parallel to *a b*, thro' a Point aſſigned, and any Parallelogram readily be deſcribed.

PROB.

PROB. II.

A Triangle abd *being given, to make a Parallelogram equal to it, which ſhall have a given Angle equal to* A.

Biſſect the Baſe of the Triangle in *c*: Make the Angle *c d e* = A, thro' the Vertex *a* draw *a e* Parallel to the Baſe *b d*. Make *a e* = *c d* and draw *a c*. So will *c e* be the Parallelogram required.



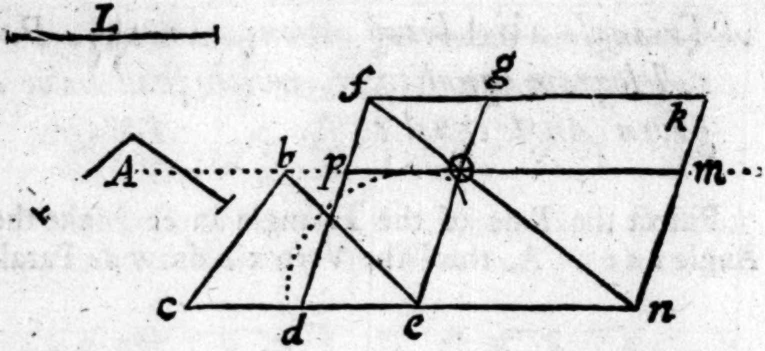
For being on but half the Baſe, and of the ſame height with the Triangle, it will be equal to it, by the 18 of this Book, and its Angle *c d e* is equal to A. Q. E. F.

PROB. III.

On a Line given as L, *to make a Parallelogram, equal to a given Triangle* cbe, *and having an Angle equal to an Angle given, as* A.

Make the Parallelogram *d o* equal to the Triangle, and having its Angle *e* = A, by Problem the laſt.

Then produce *p o* till *o m* become equal to L,and draw out *d e* till *e n* be alſo equal to L. Then draw the Di-



agonal *n o*, producing it till it meet with *d p* alſo produced to *f*. Then draw *f k* = *d n*, and *n k* = *d f*; and That will compleat the Parallelogram *f n*, in which the Complement *g m* will be equal to *p e* (3. 13) which is equal to the Triangle *c b e*. Q. E. F.

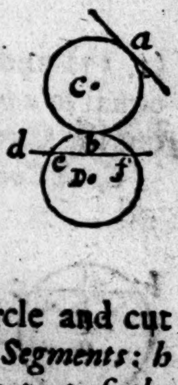
And thus 'tis eaſie to make a Parallelogram equal to any Right-lined Figure given: By reducing that Figure into Triangles, &c.

ELEMENTS
OF
GEOMETRY.

BOOK IV.

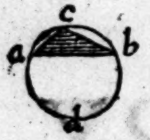
Of a Circle.

1. A Line is ſaid to *Touch (or to be a Tangent to)* a Circle, when tho' produc'd both ways from the Point of Contact, it will only touch it, and not *cut* or enter within it. Thus the Line *a* touches the Circle C, as that Circle C doth the Circle D; but *d* enters within the Circle, and *cuts* it, *and is call'd a Secant.*

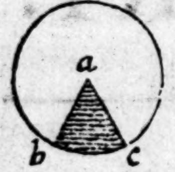


2. If a right Line enter within a Circle and cut it into two Parts, thoſe Parts are called *Segments*; *b* is a leſs Segment and D a greater: That *part of the*

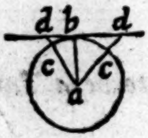
Line cutting the Circle (*and which is within it*) is called a *Chord* as *e f*. And the Parts of the Circle (*or rather Circumference*) cut off, are called *Arks*: The Chord with the Ark makes two mixt Angles as *e* and *f*, and they are call'd *Angles of a Segment*.



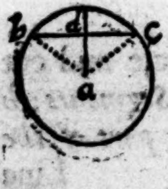
3. If you take a Point as *c*, in the Ark of any Segment, and from thence draw two Lines *c a* and *c b* (*to the ends of the Chord*) they ſhall make an Angle *a c b*; which is call'd *an Angle in a Segment*: And that Angle *a c b* is ſaid to *inſiſt* or *ſtand* on *a b d*, the Ark of the other Segment below.



4. A Sector of a Circle is a mixt Triangle comprehended between two Radii, *a b*, *a c*, and the Ark of the Circle *b c*: 'tis mark'd in the Figure by being ſhaded.

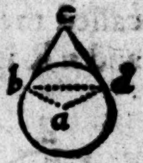


5. If at the end of any Radius, or Semidiameter, *a b*, you draw a Perpendicular, as *d b*, it ſhall touch the Circle but in one Point. And all the Points of the Line *b d* ſhall be without the Circle. *v. g.* I ſay the Point *d* (or any other aſſignable) is without: For if you draw the Line *a d* from the Center, *and that* ſhall cut the Circle in the Point *c*, that Line *a d* will be longer than *a b*; (2. 17) and conſequently longer than *a c* which is equal to *a b* (1. 14). Wherefore the Point *d* is without the Circle. Q. E. D.

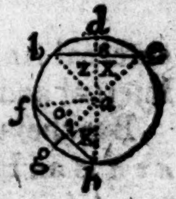


6. A *Chord*, as *b c*, is divided into two equal Parts (or Biſſected) by a Perpendicular *d a*, drawn from the Center *a*. For the Triangle *a b c* is an *Iſoſceles*, becauſe *b a* is equal to *c a* (1. 14) and therefore the Perpendicular *a d* Biſſects the Baſe *b c* (2. 16). The Ark *b c* is alſo by this means Biſſected.

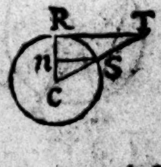
7. Two Tangents, *c b* and *c d*, drawn from the ſame Point without a Circle, are equal one to another. For, draw from the Center to the Points of Contact, *b a* and *a d*. Then will thoſe Lines be Perpendiculars to the Tangents (by 4. 5.) Then if you draw alſo the Line *b d*, the Angle *a b d* will be equal to *a d b* (2. 15.) Wherefore if from the Right and (*conſequently*) equal Angles *c b a* and *c d a*, you take away the equal Ones *a b d* and *a d b*, the remaining Angles *c b d* and *c d b* will be equal: Wherefore their oppoſite Sides muſt alſo be equal *by the Converſe of* (2. 15.) That is, *c b* is equal to *c d*. Q. E. D.



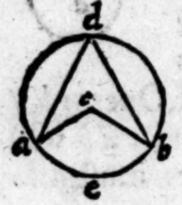
8. Equal Chords as *b c* and *f h*, do cut off equal Segments *b d c* and *f g h*. And the Perpendiculars *a e* and *a i*, drawn to them from the Center, are alſo equal, as is eaſily proved; (ſaith *Pardic*, *But he gives us no Demonſtration.*) *Yet 'tis plainly thus proved: The Chords and Arks are both Biſſected by the Perpendiculars* (4. 6.) *And therefore the Sectors* c a d, d a b, f a g *and* g a h, *muſt be all equal; as alſo will all the Triangles* x, z, o *and* k, *be* (2. 11.) *Therefore their Doubles will alſo be equal,* i. e. *The Sector* b a c *will be equal to* f a h: *And the Triangle* b a c *to the Triangle* f a h. *And if theſe laſt Triangles are taken from the equal Sectors* h a f *and* b a c, *the Segments* b d c *and* h g f *muſt remain equal. That the Perpendiculars are equal, is plain from the Equalities of the Triangles* z *and* o, *or* X *and* K.



9. Let there be a Semidiameter R *c*, and a Perpendicular (*to it without the Circle*) RT, another Line cutting the Circle in S, and a Perpendicular (*let fall from thence*) to the Radius R C in *n* (*a Point within the Circle*). All theſe Lines have Artificial

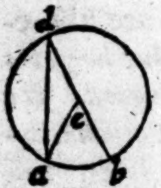


cial Names. The Line TR is called the Tangent of the Ark RS (*which ſuppoſe* 30°.) TC is called the Secant of the ſame Ark of 30°, and the Line Sn is called the (Right) Sine of the ſame Ark. RC is by ſome called the Whole Sine, but moſt uſually the Radius. and nR is called the Verſed Sine of the ſame Ark.



10. If in the Circumference of a Circle, you take two Points, as a and b, and from thence draw two Lines to the Center c, and two others to any Point, as d in the Circumference; they will make two Angles, of which acb is called an *Angle at the Center*, and adb an *Angle at the Circumference*.

11. The Angle at the Center acb is always double to one at the Circumference adb (*inſiſting with it on the ſame Ark* ab.)

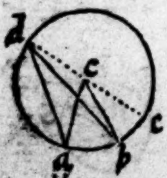


Of which there are three Caſes.

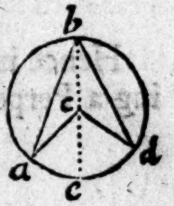
I. If one of the Lines, as db, paſs thro' the Center c, then 'tis plain the external Angle acb (2. 10.) will be equal to both the Internal and Oppoſite Ones a and d taken together.

But the two Angles d and a are equal, becauſe acd is an Iſoſceles Triangle, whoſe Side ac is equal to cd (2. 15.) Therefore the Angle c at the Center being equal to both, is double of either alone: That is, double to d. Q. E. D.

II. If neither of the Lines db, da, (*which form the Angle at the Circumference*) paſs thro' the Center c: (*But fall both on the ſame ſide of the Diameter*) Let the Diameter dce be drawn. Then will the whole Angle ace (*at the Center*) be double to the Angle ade (at the Circumfe-

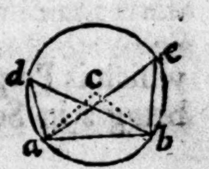


cumference) by what was proved in the firſt Caſe. Alſo the Angle *b c e* is double to *b d e*, by the ſame. Wherefore if from the Angle *a c e*, we take away that *b c e*, and from the Angle *a d e*, which is the half of *a c e*, we take away *b d e*, which alſo is the half of *b c e*, the remaining Angle *a d b* muſt be juſt the half of *a c b*. For 'tis as plain as an Axiom, that if one Quantity be double to another, and you take away from the bigger juſt the double of what you take from the other, the remainder of the Bigger muſt be double to the remainder of the Leſſer.



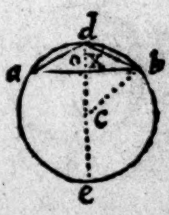
III. If the Diameter fall between the Lines forming the Angle at the Circumference. Then will, as before, the Angle *a c e* be double to *a b e* (*by Caſe* 1. *of this*) and the Angle *e c d* will be double to *e b d* by the ſame; therefore the whole Angle *a c d* muſt be double to *a b d*. (*So that in all Caſes, the Angle at the Center is double to one at the Circumference if they both ſtand on the ſame Ark, or (which is all one) are in the ſame Segment.*

12. All Angles (*in the ſame Segment or*) Inſiſting on the ſame Ark *a b* are equal, let them Terminate in any part of the Circumference whatſoever.

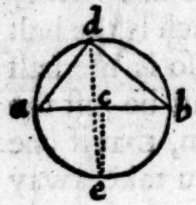


For the Angle *a d b* will be equal to *a e b*, becauſe each is the half of the Angle at the Center *a c b* (4. 11.)

13. An Angle at the Center *b c e* ſtanding on half of the Ark *a e b*, is equal to the Angle *a d b* at the Circumference ſtanding on the whole Ark. *For* c *is equal to twice* x; (by 4. 11.) *and* x *is equal to* o, *that is to half* a d b (4. 6, and 4. 8.) *Wherefore* c *is equal to* a d b. Q. E. D.

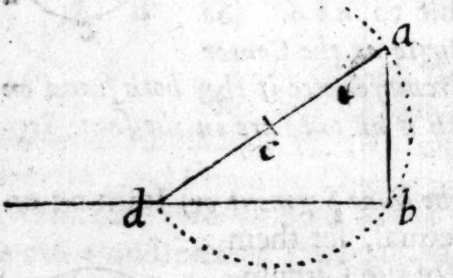


14: The Angle *a d b* ſtanding on the Semi-circumference *a e b* (*or being in the Semi-circle* adb) is a Right One. Let *c e* be drawn Biſſecting the Semi-circumference *a e b*, then is (by the *Precedent*) the Angle *a c e* at the Center, ſtanding on half a Semi-circle (*or on a Quadrant*) equal to *a d b* at the Circumference, which ſtands on twice that Ark or on a Semi-circle. But *a c e* is a Right Angle, wherefore *a d b*, (it's equal) muſt be ſo too.



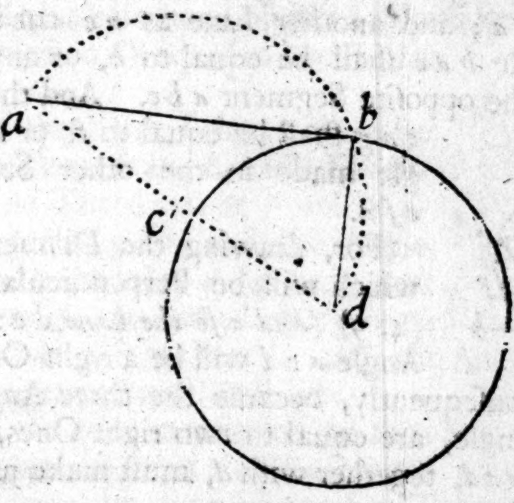
COROLLARY I.

Hence is derived the Common Practice of Erecting a Perpendicular, as *a b*, at the end of a given Line. For, opening the Compaſſes to any convenient diſtance, ſet one Point in *c*, and with the other draw the Ark *d b a*, cutting the given Line in *d*; then a Ruler laid from *d* to *c* ſhall find the Point *a*, which is Perpendicularly over *b*: For the Angle *d b a*, being a in Semi-circle, is a Right One.

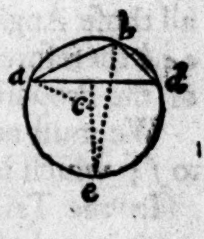


COROL. II.

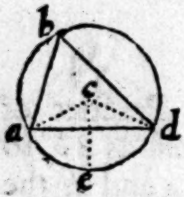
Hence alſo ariſes this expeditious Practice of drawing from a Point given as *a*, a Tangent as *a b*, to a given Circle. For joining the Points *a* and *d*, the Center of the Circle, Biſſect their Diſtance *a d* in the Point *c*: on *c*, as a Center, deſcribe the Semi-circle *a b d*: So ſhall *a b* be a true Tangent, becauſe the Angle *a b d* being in a Semi-circle is a Right One.



15. The Angle *a b d* in a Segment leſs (*than a Semi-circle*) is Obtuſe: Becauſe the Ark *a e d* being more than half the Circumference, its half, the Ark *a e*, muſt be more than 90°, therefore the Angle *a b d*, which is equal to *a c e*, (4. 13.) muſt alſo be more than 90°, that is Obtuſe.

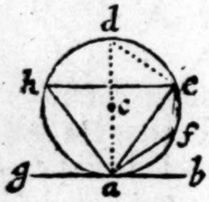


16. The Angle *abd* made in a Segment greater than a Semi-circle, is *Acute.*



For 'tis equal to the Angle *ace* (4. 13) whoſe Meaſure *ae* being the half of *aed*, an Ark leſs than a Semi-circle, muſt be leſs than 90°. And therefore *abd* is leſs than 90°. (i. e.) *Acute.*

17. If a right Line as *gb*, touch a Circle, as in the Point *a*; and another Line as *ae* cut it there. The Angle *bae* ſhall be equal to *h*, or any Angle made in the oppoſite Segment *ahe*. And the Angle *eag* ſhall be equal to *f*, or any Angle made in the other Segment, *efa*.



For, drawing the Diameter *ad*, which will be Perpendicular to *ab* (4. 9) *(and alſo the Line* de:*)* The Angle *aed* will be a right One; (4. 14) and conſequently, becauſe the three Angles of every Triangle, are equal to two right Ones, (2. 9) the Angle *ead*, together with *d*, muſt make juſt another right Angle.

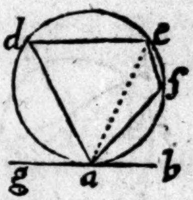
But that Angle *dae* together with *eab* doth make alſo a right One, becauſe the Radius *ca* is Perpendicular to the Tangent *ab*; wherefore *take away* ead *from both* and then *eab* will remain equal to *d*. And conſequently to *h*, or to any other Angle *in that Segment* ahe, or that ſtands on the ſame Ark *efa*. For all thoſe Angles are equal (by 4. 12). The Angle *eab* therefore, is equal to *h*: Which is the firſt part of the Propoſition.

We muſt next prove the Angle *gae* to be equal to *f*; which is the other part.

In the Triangle *afe* all the three Angles *e*, *f* and *a* are equal to two right Ones (2. 9). And the Angle *e* is equal to *fab*; by the firſt part of this Propoſition,

poſition, for *f a* may be conſider'd as cutting the Circle in the Point *a*, where *a b* touches it, and conſequently *f a b* will be equal to any Angle that can be made in the oppoſite Segment *a h d e f*; and therefore to *e*. Now the two Angles *e a f* and *f a b* (that is *e*) together with *f*, are equal to two Right Ones (2. 9), and ſo are *e a f* and *f a b* taken together with *g a e* (1. 20). Wherefore the Angle *f* is equal to *g a e*. Which was to be proved.

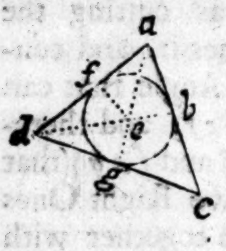
18. Every Quadrilateral Figure as *d e f a* Inſcribed in a Circle, hath its two oppoſite Angles taken together (as *d* added to *f*) equal to two Rgiht Ones.



For if thro' the Point *a*, there be drawn a Tangent as *g b*, and a Diagonal as *e a*: The Angle at *f* will be equal to *g a e* (4. 17) and the Angle *e a b* will be equal to *d* (4. 17.) And conſequently, the two Angles *g a e* and *e a b* being equal to two Right Ones (1. 20), the Angles *d* and *f* taken together muſt be ſo too.

After the ſame manner might the other two oppoſite Angles *d a f* and *d e f* be proved equal to two Right Ones; by drawing another Tangent through the Point *f*.

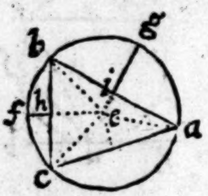
19. The Converſe of this Propoſition is alſo manifeſt; *viz*. That if any Quadrilaterial Figure have its oppoſite Angles equal to two Right Ones, it may then be inſcribed in a Circle; That is, a Circle may be made, that ſhall touch or paſs through all its four Angular Points.



20. A Rectilineal Figure is ſaid to be Circumſcribed about a Circle, when all its Sides touch the Circle without cutting it. Thus the Triangle *d a e* is Circumſcribed about the Circle *b g f*; becauſe every ſide of the Triangle touches the Circle, in *b*, *g* and *f*.

21. A Figure is ſaid to be *Inſcribed* in a Circle when all its Angles are in the Circumference of that Circle, as the Triangle *a b c* in the following Figure.

22. Every Triangle, *a b c* may be Inſcribed in a Circle; for if two Lines as *e b* and *e i* are drawn Perpendicularly Biſſecting the ſides *b a* and *c b*, they will croſs or meet each other in the point *e*, on which as on a Center, a Circle may be drawn, which ſhall paſs through *b*. And I ſay alſo, That that Circle ſhall paſs through *a* and *c*.



For I. The two Triangles *e i b* and *e i a* are equal; becauſe *i b* is equal to *i a* by the ſuppoſition, the ſide *e i* is common to both, and the Angles at *i* are Right. Wherefore the ſide *e b* is alſo equal to *e a* (2. 11.)

II. And for the ſame Reaſon *The Triangles* e h c *and* e h b *may be proved equal*, and conſequently, the ſide *e c* alſo will be equal to *e b* and to *e a*. But if thoſe three Lines are all equal, the Point *e*, where they meet, muſt be the Center of a Circle of which they are *Radii*: And therefore the Triangle is Circumſcribed by a Circle. Q. E. D.

And thus may a Circle be made to paſs through any three Points, if they be not all in a Right Line.

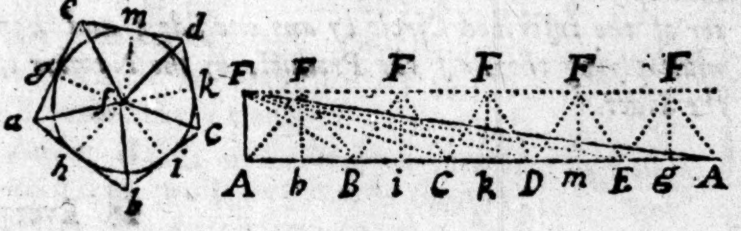
23. Every Triangle may (*have a Circle Inſcribed in it, or*) be Circumſcribed about one, *vid. Fig. in Art.* 20.

For drawing the Lines *a e* and *e d* Biſſecting the Angles *a* and *d*, and from the point *e*, where they croſs, letting fall the Perpendiculars (to the ſides of the Triangle) *e b*, *e f* and *e g*: I ſay, that if you draw a Circle on the Center *e* through *b*; that Circle ſhall touch all the Sides of the Triangle in the points *b*, *f* and *g*.

For I. The two Triangles *a e f* and *a e b* are equal, as having the ſide *a e* common, the Angles at *f* and *b* Right, and thoſe at *a* equal (by the Suppoſition:) wherefore *e b* is equal to *e f*. (2. 14.)

II. By the ſame Method, *e g* may be proved equal alſo to *e f*, (that is to *e b*) ſo that theſe three Lines being all equal, a Circle will paſs through their three extremities, of which Circle they will be Radii, and being alſo all Perpendicular to the ſides of the Triangle, the ſaid ſides are Tangents to that Circle (4. 5), and therefore do Circumſcribe it (by 4. 18.)

24. Every Polygon Circumſcribed about a Circle, is equal to a Rectangle Triangle, one of whoſe Legs ſhall be the Radius of the Circle, and the other the Perimeter (or the Sum of all the ſides) of the Polygon.



Let the Line F A be equal to the Radius *f h*, and to it at Right Angles draw the infinite Line A B C D, *&c.* out of which take A *h* equal to *a h*, *h* B equal to *h b*, B *i* equal to *b i* and *i* C equal to *i c*, &c. So that the whole Line A B C D E A, may be equal to the whole Compaſs or *Perimeter* of the Polygon *a b c d e a*. Alſo draw F F Parallel to A A, So that all the Perpendiculars F *h*, F *i*, F *k*, &c. may be equal to the Radius *f h* or *f i*, &c. 'Tis then plain, that the Triangle A F B will be equal to the Triangle *a f b* in the Polygon, and the Triangle B F C to *b f c*, and alſo C F D to *c f d*, &c. So that all theſe Triangles taken together will be equal to all theſe in the Polygon, or to the whole Polygon.

But the Triangle F A A is equal to all the five Triangles within the Parallels; becauſe drawing the Lines, B F, C F, D F, *&c.* The Triangle F A B will be equal to F A B, F B C to F B C, *&c.* (3. 16); wherefore the Triangle F A A is equal to the Polygon, which was to be proved.

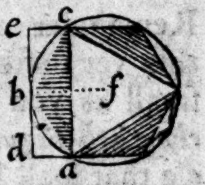
25. Every Regular Polygon is equal to a Rectangle Triangle, one of whoſe Legs is the Perimeter of the Polygon, and the other a Perpendicular drawn from the Center to one of the ſides of the Polygon. The Proof of which is the ſame as that in the precedent Propoſition; For all the Perpendiculars *f h*, *f i*, *f k*, &c. are equal *&c. See the laſt Figure.*

Wherefor the Area of every regular Polygon, is found by multiplying the Perpendicular let fall from the Center of the inſcribed Circle by any one ſide; and than multiplying the half the Product, by the Number of the Sides.

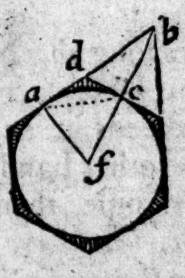
26. Every Polygon Circumſcribed about a Circle, is bigger than it; and every Polygon Inſcribed, is leſs than the Circle; As is manifeſt, becauſe the thing containing, is always greater than the thing contained.

27. The *Perimeter*, or (as ſome call it, tho' improperly) the Circumference of every Polygon Circumſcribed about a Circle, is greater than the Circumference of that Circle; and the Perimeter of every Polygon Inſcribed, is leſs.

28. If in any little Segment of a Circle, you Inſcribe an Iſoſceles Triangle, as *a b c*; ſo that *a b*, be equal to *b c*: I ſay that Triangle ſhall be greater than half that Segment. For if you draw a Tangent *e b d*, which ſhall be Parallel to *c a*; and which ſhall be as *c a* is, Perpendicular to the Radius *b f*; (4. 5) (4. 6) And then Compleat the Rectangle *a d e c*; That Rectangle will be greater than the whole Segment *a c b*: But the Triangle *a b c*, is the half of that Parallelogram (3. 18). And therefore muſt be greater than half the Segment *a b c*.



29. Let there be a Tangent *a d b*, a Secant *f c b*, a Chord *a c*, and another Tangent *c d*, I ſay, that the Triangle *d b c* is more than half the mixt Triangle *a c b*, comprehended between the Lines *a b*, *b c* and the Ark of the Circle *a c*. For in the Triangle *d b c*, the Angle *c*, being a Right one (4. 5), the ſide *d b*, is longer than *d c* (2. 17). That is, than *d a*; which is equal to *d c* (4. 7). Wherefore the Triangle *b d c* (*having a longer Baſe, but the ſame height with* a d c) muſt be greater than it; (*as may be collected from* (3. 7). And therefore it muſt be greater than the



half of the whole Triangle *a c b*. But the Triangle *a c b*, is greater than the mixt Triangle, made by the Ark *a c*, and the Right Lines, *a b* and *a c*; and therefore the Triangle *b d c* (which is more than half of *a c b*) muſt be greater than the half of the mixt Triangle *a b c*. Q. E. D.

30. From theſe two laſt Propoſitions, it follows that by multiplying the ſides of Polygons, you may make them ſo *Circumſcribed* about, or *Inſcribed* in Circles, that the difference by which the Circumſcribed exceeds, or the Inſcribed wants of the Circle, ſhall be as ſmall as you will; Becauſe if from any Quantity whatever, you take more than the half, and from the Remainder more than its half, and again from that Remainder more than its half: You may by doing this very often at laſt come to leave a Remainder as ſmall as you pleaſe; as is Self-evident. Thus (*See the 28th Figure*) After a Triangle is inſcribed in a Circle that ſhall be leſs than it by the three great Segments, you may inſcribe an Hexagon that ſhall exceed the Triangle by thoſe three Segments, but ſhall be leſs than the Circle, by the ſix little Segments, that are left white in the Figure.

But thoſe ſix white Segments taken together, do not contain ſo much Space as the half of the three former Shaded ones, (4. 28.) After this you may alſo Inſcribe a *Duodecagon*; which will be leſſer than the Circle by 12 ſmaller Segments; which 12 Segments will ſtill be leſs than the half of the ſix Segments of the Hexagon: And thus may you by increaſing the Number of ſides of the Polygon, leſſen the Difference by which the Circumſcribing Circle exceeds it, as much as you pleaſe. So likewiſe on the other hand, you might have firſt Circumſcribed a Triangle, then an Hexagon, and then a Duodecagon, &c. (*and have made*, that way, *the Differenee between the Circumſcribing Polygon and the Circle, as ſmall as you would.*)

31. Every Circle is equal to a Rectangle Triangle; one of whoſe Legs is the Radius, and the other a Right Line equal to the Circumference of the Circle. For ſuch a Triangle will be greater than any Polygon Inſcribed, and leſs than any Polygon Circumſcribed, (by 24, 25, 26 and 27 of this fourth Book.) And therefore muſt be equal to the Circle.

For ſhould it be greater than the Circle, be the exceſs as little as it will, a Polygon may be Circumſcribed, whoſe Difference from the Circle ſhall be yet leſs than the Difference between that Circle and the Rectangled Triangle: And that Polygon will be leſs than the Triangle; which is Abſurd. And if it be ſaid that this Rectangled Triangle is leſs than the Circle; an Inſcribed Polygon may be made, which ſhall be greater than that Triangle, which is impoſſible.

“ This kind of Demonſtration which we here uſe, and which is called *Reductio ad Abſurdum ſive ad Impoſſibile*, is one of the fineſt Inventions of the Ancients: And on it is founded all the *Geometry* of *Indiviſibles*; ſo that I cannot but much wonder ſome of our Modern Authors ſhould reject it as indirect and deficient. But if we muſt arrive to ſuch a point of Niceneſs, that we can't bear any Demonſtration, unleſs it be Direct and Poſitive; 'tis eaſie enough to give this before us ſuch a turn, as ſhall render it Regular and Direct.

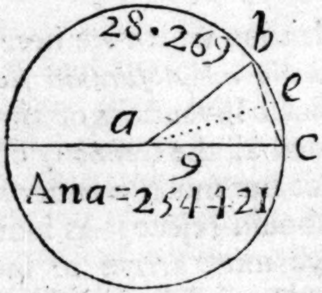
“ For this cannot but be admitted as a Principle; *That if two determinate Quantities* a *and* b *are ſuch, that every other imaginable Quantity, which is greater or leſs than* a, *is alſo greater or leſs than* b; *Theſe two Quantities* a *and* b *muſt be equal.* And this Principle being granted, which is in a manner ſelf-evident, it may directly be proved that the Triangle (before mentioned) is equal to the Circle: Becauſe every Imaginable inſcribed Figure, which is

“ leſs than the Circle, is alſo leſs than the Triangle:
“ And every Circumſcribed Figure greater than the
“ Circle, is alſo greater than the Triangle.

This is that which is called the Quadrature of (or Squaring) the Circle, which conſiſts in finding a Square, Triangle, or any other Rectilineal. Figure exactly equal to a Circle. And this would eaſily be done, could we find a Right Line equal to the Circumference; as is plain from this laſt Propoſition. But ſuch an Equality is not to be found Geometrically.

To find ihe Area of a Circle.

Since the Circle is equal to a Right-angled Triangle, whoſe Baſe is the Radius, and the Perpendicular a Line equal to the Circumference; half the Product of the Radius into the Periphery, will give the Area of the Circle.

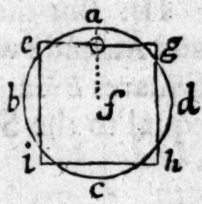


In Practice, therefore ſay, either as 7 : to 22 :: So is the Diameter in Inches, equal Parts, &c. to the Circumference or more nearly and without Diviſion, ſay, as 1000 is to 3141 :: So is the Radius of any Circle in Inches (ſuppoſe 9 Inches) to 28.269, which therefore will be the Semi-circumference: And this multiplied by 9 the Radius, gives 254.421, for the Area required.

For the Area of a Sector or Segment of any Circle.

Since a Circle may be conceived as an Aggregate of an infinite Number of Iſoſceles Triangles, whoſe common Vertex is the Center: Any Portion of the Periphery, as *b c*, being conſidered as a ſtrait Line, and the Perpendicular *a e* let fall, the Area of the Sector muſt be half the Product of the Ark *b c* into the Radius *a e*; and if from the Sector you take the Area of the right Lined Triangle *a b c*, there will remain the Area of the Segment *b e c*.

32. If a right Line could be diſpos'd into the Form of the circumference of a Circle, it would contain more Space than any other Figure, or Regular Polygon whatſoever: Suppoſe the circumference of the Circle *a b c d*, to be diſpos'd into the Form of a Square, or into any other Regular Polygon: So that all the Sides *e g*, *g h*, *h i* and *i e* together may be equal to the circumference *a b c d*: I ſay, the Circle is greater than that Square. For the Circle is equal to a Rectangle Triangle one of whoſe Legs is the Radius *f a*, and the other the circumference. And the Polygon, is equal alſo to ſuch a Triangle, one of whoſe Legs is the ſame circumference *a b c d*, or the Sum of the Sides *g e i h*; and the other Leg is the Line *f o* (4. 25.) But as the Line *f o* is leſs than the Radius *f a*, ſo the ſecond Triangle, which is equal to the Polygon, muſt be leſs than the firſt which is equal to the Circle, and therefore the Square or Polygon muſt be leſs than the Circle, which was to be Demonſtrated.



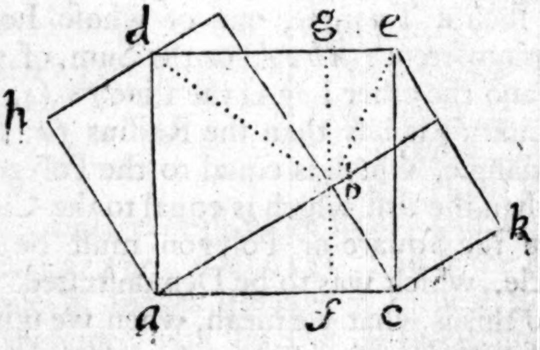
“ And this is what we mean, when we uſually ſay, “ that of *Iſoperimetrical Figures* (or which have equal “ *Perimeters* or *Circumferences*) the greateſt is the “ Circle.

Before we go to Solids, I thought it proper to give the Learner here, this moſt Noble Theorem of *Pythagoras*; becauſe, tho' it be indeed Demonſtrated in the ſixth Book, yet nearly after *Euclides* manner, it may alſo be done here; Thus, In every Right angled Triangle as *a b c*, The Square of the Hypothenuſe *a c*, is equal to the Sum of the Squares of Legs *a b* and *b c*; For,

I. The Square of *c a*, is equal to the two Rectangles *d f* and *f e*.

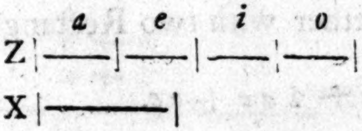
II. The Rectangle *d f* is double of the Triangle *a b d* being of the ſame Baſe and Altitude; and the Rectangle *f e* is for the ſame reaſon, double of the Triangle *b e c*. (by 3. 18.)

III. But thoſe Triangles, being of the ſame Baſe and Altitude with, will be equal alſo to one half of the Squares *h b* and *b k*: Wherefore the Square of *a c* is equal to the Sum of the Squares of the Legs.



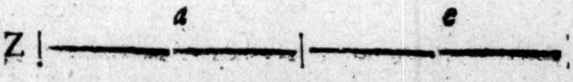
I have here added alſo the Subſtance of the ſecond Book of *Euclide* about the Power of Lines, *&c.* And I would Adviſe the Young Geometrician before he proceeds any farther (and if not done already) to begin the Study of Algebra; a little of which will be of excellent uſe to him in facilitating the Demonſtrations in Geometry, and in preparing the Mind and enuring of it to Abſtraction, before he come to the Doctrine of Proportion. And the Knowledge of the four firſt Rules of *Addition*, *Subtraction*, *Multiplication* and *Diviſion*, in *Integers* and *Fractions* will be ſufficient to enable him to underſtand the following Propoſitions. As alſo thoſe moſt uſeful Ones, which he will find added (in this Edition) in all the following Books of theſe Elements.

I. If there be two Lines z and X; one of which, as z, is divided into any Number of Parts, as into $a + e + i + o$. The Rectangle under the two whole Lines $z\,x$, is equal to the Sum of all the Rectangles made by x multiplied into the Parts of z,



That is $ZX = Xa + Xe + Xi + Xo$, This is ſo plain, it needs no Proof.

II. If a Right Line, as Z, be divided into two parts $a + e$. The Rectangles made by the whole Line, and both its Parts are equal to the Square of the whole Line.

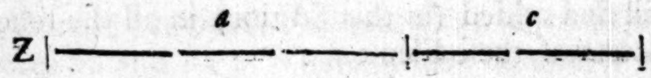


That is, $za + ze = zz$.
For $za = aa + ae$.
And $ze = ae + ee$.

That is,
$za + ze = aa + 2ae + ee =$ Q. $a + e$ Q. E. D.

III. Let the Line Z be cut into $a + e$; then ſhall the Rectangle under the whole Line (Z) and the Part (a) be equal to the Square of that Part a, together with the Rectangle made by the two Parts a and e.

That is, $Za = aa + ae$.

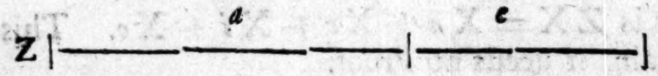


For $Z = a + e$

And $\overline{a + e} \times a = aa + ae$. Q. E. D.

IV. The Square of any Line, as Z, divided into any two Parts a and e, is equal to both the Squares of thoſe Parts, together with two Rectangles made out of thoſe Parts.

That is, $zz = aa + 2ae + ee$,



Multiply $a + e$ by its ſelf, and the thing is plain.

$$\begin{array}{l}
a+e \\
a+e \\
\hline
aa+ae \\
\quad\;\; +ae+ee \\
\hline
aa+2ae+ee \\
\hline
\end{array}$$

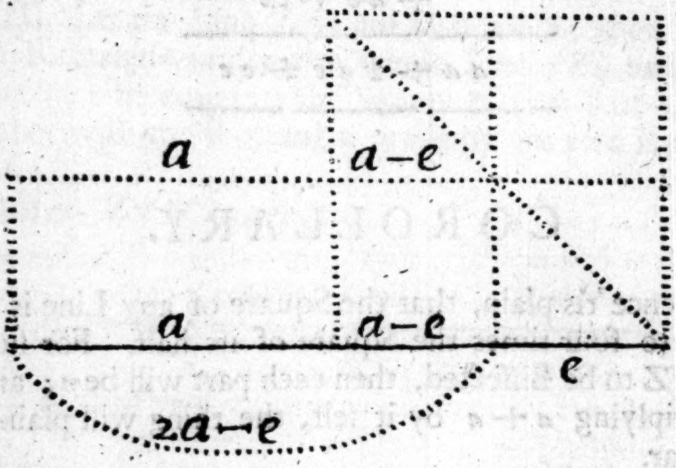
COROLLARY.

Hence 'tis plain, that the Square of any Line is equal to four times the Square of its half. For ſuppoſe Z to be Biſſected, then each part will be a; and multiplying $a+a$ by it ſelf, the thing will plainly appear.

$$\begin{array}{l}
a+a \\
a+a \\
\hline
aa+aa+aa+aa=4aa
\end{array}$$

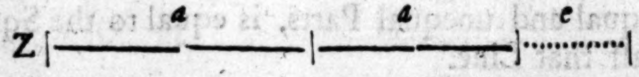
V. If a Line be divided into two Parts equally, and into two other Parts unequally, the Rectangle under the unequal Parts, together with the Square of (the intermediate Part) the Difference between the equal and unequal Parts, is equal to the Square of half that Line.

Let the whole Line be $2a$, then each Part will be a. Let the leſſer unequal Part be e, then the Intermediate Part will be $a - e$, and the greater unequal Part will be $2a - e$; which multiplied by



e, produces $2ae - ee$; To which adding the Square of the Difference, or intermediate Part $a - e$, which is $aa - 2ae + ee$, the Sum will be only aa, the Square of half the Line.

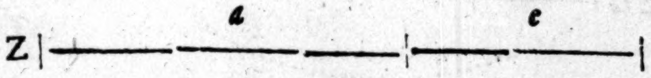
VI. If a Line be Biſſected, and then another Right Line be added to it, the Rectangle or Product of the whole Augmented Line, multiplied by the Part added, together with the Square of the half Line, is equal to the Square of the half Line and part added, conſidered as one Line.



Let the firſt Line be $2a$, and the Part added e, then the whole will be $2a + e$; which multiplied by e, produces $2ae + ee$, and the Square of half the

Line aa being added to it, it will be $2ae + ee + aa$, which is equal to the Square of $a + e$, by Prop. 4.

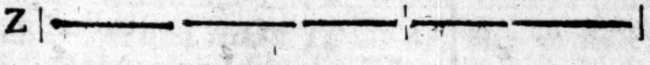
VII. If a Quantity or Line be divided any how into two Parts, the Square of the Whole added to the Square of one of the Parts, ſhall be equal to two Rectangles contained under the whole Line and that Part, added to the Square of the other Part.



Let a be one Part, and e the other. The Square of the whole and of the leſſer Part e, makes $aa + 2ae + 2ee$. Then if the whole $a + e$ be multiplied twice by e, it will produce $2ae + 2ee$; and if to this be added the Square of the other Part aa, the Sum will be

$aa + 2ae + 2ee$, equal to the former.

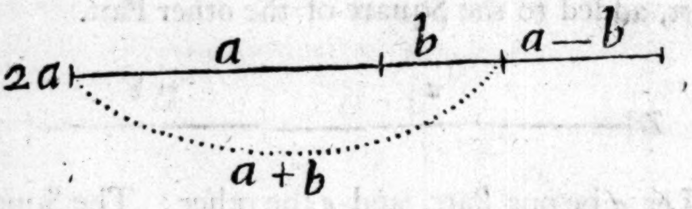
VIII. If a Line be cut any how into two Parts, the Quadruple Rectangle under the whole Line and one of the Parts, added to the Square of the other Part, is equal to the Square of the whole and the other Part added to it, as if it were but one Line.



Let the whole Line be $a + e$, then four times that multiplied by e (or the Quadruple Rectangle under that and e) will be $4ae + 4ee$; to which adding the Square of the other Part aa, the Sum will be $aa + 4ae + 4ee$.

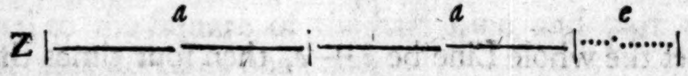
And if you ſquare $a + 2e$, which expreſſes the whole Line, with e added to it, the Product will be the former Sum of $aa + 4ae + 4ee$.

¶ IX. If a Line be biſſected, and alſo cut into two other unequal Parts, the Sum of the Squares of the unequal Parts will be double to the Sum of the Squares of the half Line, and of the Difference between the two unequal Parts.



Let the whole Line be $2a$; and the Difference between the equal and unequal Parts, b; then the greater unequal Part will be $a+b$, and the leſſer $a-b$: The Sum of the Squares of the unequal Parts will be $2aa+2bb$, which is double to the Square of half the Line added to the Square of the Difference. *Q. E. D.*

X. If a Line be biſſected, and then another Line added to it; the Square of the whole encreaſed Line, together with the Square of the Part added, is double the Sum of the Squares of the half Line, and of the half Line and Part added, taken as one Line.

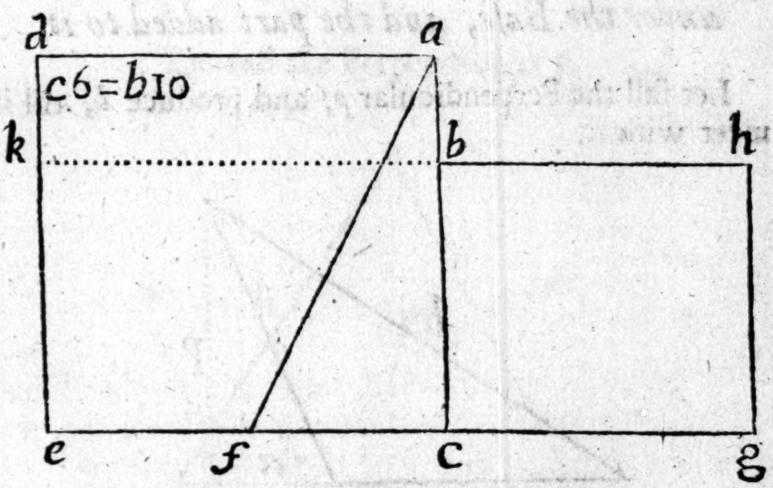


Let the whole Line be $2a$, and the Part added e; then the whole encreaſed Line will be $2a+e$; and the half Line and Part added will be $a+e$. The Sum of the Squares of $2a+e$, and of e, is $4aa+4ae+2ee$; which is plainly double to aa, and $aa+2ae+ee$ added together. *Q. E. D.*

This Problem is alſo of frequent uſe.

PROBLEM.

To divide a Line ſo, as that the Rectangle under the whole Line a c, *and one Segment* a b, *ſhall be equal to the Square of the other* b c.



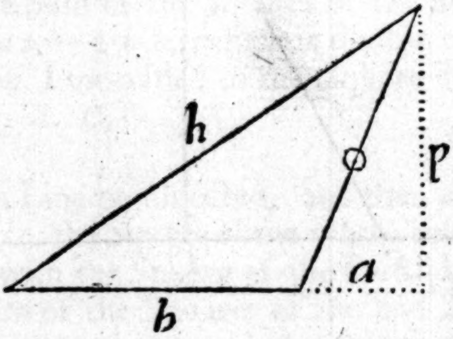
On ac make the □ cd, whoſe Baſe ec biſſect in f and draw af; make $fg = af$, and compleat the □ bg, producing bh to k: Then is ac truly divided in b; for the Line ec being biſſected in f, and the part cg added to it the (*by* Prop. 6. *of the Power of Lines*) Rect. $kg + fcq = fgq = faq = acq + fcq$: Wherefore taking fcq from both, the Rect. $kg = acq$, and taking the Rect. kc from both: the Rect. $db =$ □ bg; that is Rect. $cab = bgq$. Q. E. F.

N. B. *This is called dividing a Line according to Extream and Mean Proportion; which Proportion cannot be expreſs'd in Numbers.*

PROP. I.

In an Obtuſe-Angled Triangle, *the Square of the Side-ſubtending the Obtuſe Angle, exceeds the Sum of the Squares of the other two Sides by the double Rectangle,* (2 b a) *under the Baſe, and the part added to it.*

Let fall the Perpendicular *p*, and produce *b*, till it meet with it.



DEMONSTRATION.

1. $hh = bb + 2ba + aa + pp$.
2. And $oo = pp + aa$.
3. But $bb + oo = bb + aa + pp$.

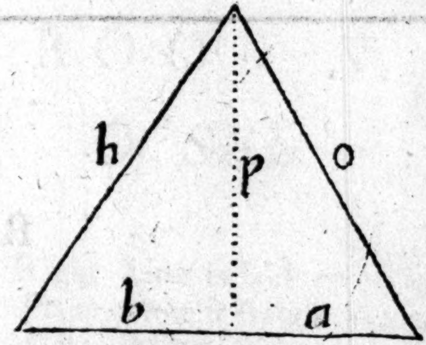
Wherefore hh exceeds the laſt Step by $2ba$. Q. E. D.

PROP.

PROP. II.

In an Acute-angled Triangle, *the Square of the Side* (h) *ſubtending an* Acute-angle, *is leſs than the Sum of the Squares of the other two Sides, by double the Rectangle under the whole Baſe,* (b + a) *and the Segment of the Baſe* (a) *which is next to the* Acute-angle.

Let fall the Perpendicular *p*.



DEMONSTRATION.

1. $hh = bb + pp$.

2. $oo = pp + aa$.

3. Q. $\overline{b + a} = bb + 2ba + aa$.

4. $bb + pp + 2aa + 2ab$, is the Sum of the Squares of the Legs.

Wherefore hh is leſs than that by $2aa + 2ab$, which is plainly equal to the Double Rect-angle under the whole Baſe, and the Part a.

BOOK

ELEMENTS OF GEOMETRY.

BOOK V.

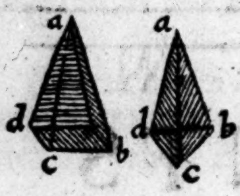
Of Solids.

1. A Right Line is ſaid to be *Right* upon a Plane, when it ſtands on it at Right Angles, juſt like a Pillar on the Ground, and is inclined no more to any one ſide of the Plane, than to the other.

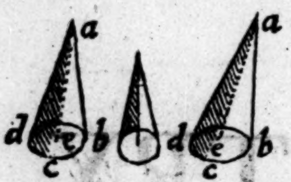
2. Two Planes are Parallel to each other, when all the Perpendiculars that can be drawn between them, are equal. (That is, *when they every where are equally diſtant.*

3. One Plane is *Right* or Perpendicular to another Plane, when like a well-made Wall, it inclines and leans on one ſide no more than it does on the other.

4. A *ſolid Angle* is made by the meeting of three or more Planes, and thoſe joining in a Point: like the Point of a Diamond well cut.

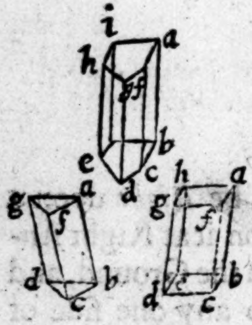


5. If we imagine a Line as *a b*, fixt above in the Point *a*, to be moved along the Sides of any Polygon *d b c*; that Line by its Motion ſhall deſcribe a Figure that is call'd a *Pyramid*.



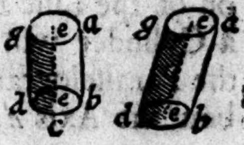
6. The *Polygon* is called the *Baſe* of the Pyramid.

7. If a Line faſtned, as before, move round a Circle, as *d b c*, it will deſcribe a Cone; and the Circle is its Baſe. And a Line drawn from the Center *e* to *a*, is call'd its *Axis*.



8. If a Line *a b* move uniformly about two Polygons *g f a* and *d c b*, which are every way equal, having their Sides and Angles mutually Parallel and correſponding exactly to one another, as *a f* to *b c*, *f g* to *d c*, &c. then that Line ſhall by its Motion deſcribe a Figure which is call'd a *Priſm*, and the Polygon is its Baſe.

9. If the Baſe of a Priſm be a Parallelogram, then that Priſm is call'd a Parallelopiped.



10. If a Line *a b* moved uniformly round two equal and Parallel Circles, it ſhall deſcribe or generate a *Cylinder*.

11. The Line joyning the Centers *e e*, in the two Baſes, is call'd the Axis.

“ There is no need of conceiving two Baſes, E-
“ qual, Parallel and Oppoſite, for the Geneſis of
“ Priſms and Cylinders. For they will be deſcrib'd as
“ well by Imagining a Line moving round the Cir-
“ cumference of any plane Figure with a Motion al-
“ ways Parallel to its ſelf in its firſt Poſition. As if
“ *a b* be ſuppoſed to be carried round any of the Ba-
“ ſes *d c b*, keeping always the ſame Angle with the
“ Plane, which it firſt had, it will deſcribe a *Trian-*
“ *gular*, *Quinquangular*, or *Circular Priſm*, according
“ to the Figure of the Baſe. And the upper End of
“ the Line will deſcribe a Baſe, (as you may call it)
“ at the Top, equal and parallel to that below.

COROLLARY.

The Solid Content of all Iſoſceles, Priſms and Cylinders (as alſo, of all Parallelopipeds), is had by Multiplying their Height into the Area of their Baſe.

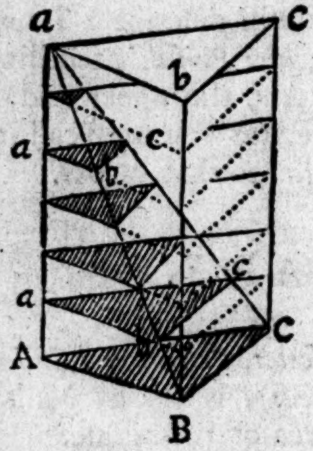
And if they are Scalenous Priſms or Cylinders, by Multiplying the Baſe by the perpendicular Altitude.

But after all, this Geneſis of Priſms and Pyramids, of Mr. *Pardie*'s, reſpects only their Surfaces. And therefore, the moſt proper way to conceive the Geneſis of all Kinds of Priſms, is to imagine a Triangle, Quadrilateral Figure, or Polygon, or the plane of a Circle to be moved in a Poſition always Parallel to it ſelf; as ſuppoſe from *h* to *e*, or from *g* to *d* (in the preceeding Figures) according to the Direction of the Line *h e* or *g d*. Or according to *Euclide*, a Cylinder will be generated by the Revolution of the Parallelogram *g e d e* (*See Fig. in Art.* 8.) round about the Axis *a e*.

COROLLARY.

And from hence (as was obſerved before of Lines) 'tis plain that equal Surfaces moved uniformely over equal Places or Intervals, will deſcribe or generate equal Solids.

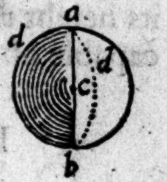
And as for the Geneſis of Pyramids, ſuppoſe the Triangle *a b c*, to move downwards from the Top of a Plane Angle, determined by the two Planes *a* A B, *a* A C: Let this Motion be always Parallel to it ſelf, and let the Angular Point of the moving Triangle *a*, be ſuppoſed always to keep in the Line *a* A.



'Tis plain, as this Triangle moves farther downwards, it will ſtill get more and more within the Solid Angle, and at laſt will come to be all of it within it, and to lie in the Poſition A B C, which will be the Baſe of a Triangular Pyramid, whoſe Vertex is at *a*.

The ſame Triangle *a b c*, will alſo, by its Motion, deſcribe another Pyramid, whoſe Baſe ſhall be the Parallelogram *b c* B C, and its Vertex *a*, as before.

13. If a Semi-circle *adb* be turned quite round on its Diameter *ab*, it will deſcribe a Sphere or Globe, whoſe Axis will be *ab*, and its Center *c*, the ſame with the Semicircle. Every Line paſſing through the Center *c*, and terminated at each end by the Surface of the Sphere, is called a Diameter, and may be call'd an Axis.



14. All Lines drawn from the Center *c* to the Surface, are call'd Radii, and are all equal to one another.

To find the Surfaces of Solids.

I. *For all Priſms, Parallelopipeds and Cylinders.*

Find the Perimeter of the Baſe (which in Practice is done by girting it with a String) and Multiply that by the Perpnedicular height, the Product is the Surface without the Baſe, (*i. e.* without the top and bottom Planes) and the Baſes may be found by the Rules given in *Plain Menſuration*: The Reaſon of which is, becauſe a Rectangle of that Form and Dimenſions will juſt cover the outſide of the Body.

II. *For Pyramids and Cones.*

The Surface of a Pyramid, is only an Aggregate of Triangles, which therefore muſt be found ſeverally, and then added up into one Sum.

The Surface of Scalenous Cones cannot be found exactly; but for Right Ones Multiply the Circumference of the Baſe, by half of the Side of the Cone, the Product is the Area of the Convex Surface. Becauſe the Curve Surface of a Cone is equal to a Triangle, whoſe Baſe is the Periphery of the Baſe, and its height the ſide of the Cone; ſuch a Figure being capable of exactly covering it.

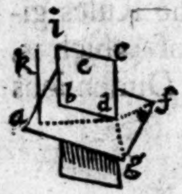
III. *For the Surface of the Sphere.*

Multiply the Diameter by the Periphery of any great Circle, or by ſuch a Circle as hath the Diameter of the Sphere for its Diameter, the Product is the Surface. As appears from what will be prov'd below after *Art.* 34.

IV. *The Surfaces of the five Regular Bodies, is eaſily had by the Principles of Plain Menſuration.*

15. Two Right Lines if they meet ſo as to cut or croſs each other, are in the ſame Plain: Wherefore all the Angles and Sides of every Triangle are in the ſame Plane.

16. If two Planes *e b d* and *a g f* cut or Interſect one another, they ſhall do ſo in a Right Line, as *b d*; which is call'd their common Section.



17. If a Right Line *d c* be Perpendicular to two Lines *d f* and *d g*, which are in the ſame Plane, that Line is alſo Perpendicular to that Plane.

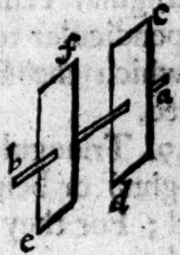
18. If a Right Line *d c* be Perpendicular to three Right Lines *d f*, *d g* and *d a*, they are all three in the ſame Plane.

19. If two Lines *d c*, *b i* are Perpendicular to the ſame Plane *f g a*, they will be Parallel to one another.

20. If two Lines *d c*, *b i* are Parallel, and you draw another Line, from any point in one to the other, as *b d*, thoſe three will be all in the ſame Plane.

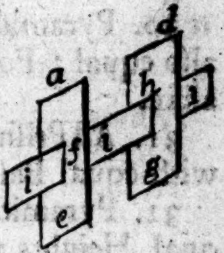
21. If two Lines *d c*, *b i* are Parallel to a third *a k*, though that third Line be not in the ſame Plane with them, yet they ſhall be Parallel to each other.

22. If a right Line *a b* be perpendicular to, (*or make any other equal Angles with*) two Planes *f e* and *c d*, thoſe Planes are Parallel.



23. If two Parallel Planes, *d h g* and *a f e* are cut by a third *i i i*, the common Sections *f e* and *h g* are Parallel.

24. If a Solid Angle be made by three Plane Angles, any two of thoſe are always greater than the third.



All theſe Propoſitions are ſo manifeſt to one that will but conſider them with a little Attention, that 'tis needleſs to ſtay to Demonſtrate them; (And indeed the Solemn and Regular Demonſtration of a thing Plane in its ſelf, always makes it more Obſcure.)

25. The *Plane Angles*, concurring to make a *Solid* One, taken all together, are always leſs than four Right Ones. For if they ſhould make four Right Angles, they would form a Plane and not an Angle. Wherefore, that they may make a Solid Angle, they muſt be leſs than four Right Ones.

'Tis a very good way in order to gain a clear Idea of Solids and their Angles, to make the Regular Bodies out of thick Paper or Paſt-board, and after the Deſcription of every Body, you will ſee the Figure, which being folded up together, will expreſs the Solid.

26. In all Parallelopipeds, the Oppoſite Planes are equal; as is eaſie to conceive (from 5. 9.)

27. All Parallelopipeds having equal Baſes (and Heights) or being between the ſame Parallels are equal, for they are equal Aggregates of equal Parallelograms. (3. 14.)

28. Every Parallelopiped is divided into two equal Triangular Priſms, by a Diagonal Plane, which is Perpendicular to its Baſe: For every Parallelogram of which the Figure is compoſed, is equally Biſſected.

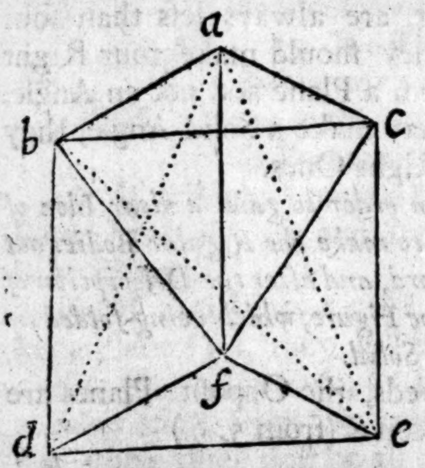
29. Triangular Priſms, having equal Baſes (and Heights) or being between the ſame Parallels, are equal; For they are equal Aggregates of equal Triangles.

30. Pyramids having equal Baſes and Heights are alſo equal; For they are all ſuppoſed to grow Taper alike.

31, All Priſms in general, all Cylinders and Cones, with equal Baſes and Heights, are equal.

32. Pyramids and Cones on equal Baſes, and of equal Heights with Priſms and Cylinders, are one third of ſuch Priſms and Cylinders.

In a Triangular Priſm, and Pyramid of the ſame Baſe and Altitude, it is thus prov'd.



The Quadrangular Pyramid *acefb* is divided into two equal Triangular Ones, by the Triangular Plane *fbc*, and the Pyramid *fcab*, is the very ſame with *bacf*; and this is equal to the Pyramid *dfea*: As having an equal Baſe and the ſame Altitude with it, and therefore the whole Priſm is divided into three equal Pyramids, And ſince all Multangular Priſms can be divided into Triangular ones, and that a Cylinder is only a Multangular Priſm of Infinite Sides, the Propoſition is Univerſally true; *That Pyramids and Cones*, &c.

N. B. A piece of Cork or Wood, in the form of a Triangular Priſm, may be cut into three equal Pyramids.

COROLLARY I.

Hence the way of finding the Solidity of a Pyramid or Cone is diſcover'd, viz. *to Multiply the Baſe by* $\frac{1}{3}$ *of the Perpendicular Altitude.*

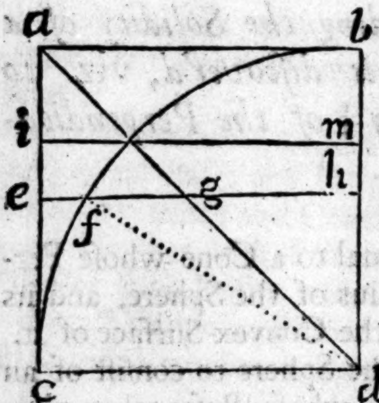
33. Every Sphere is equal to a Cone whoſe Perpendicular Axis is the Radius of the Sphere, and its Baſe a Plane, equal to all the Convex Surface of it.

For you may conceive the Sphere to conſiſt of an infinite Number of Cones, whoſe Baſes taken altogether compoſe the Surface, and whoſe Vertex's meet altogether in the Center of the Sphere: Juſt as a Circle may be imagined to be compoſed of an infinite Number of Iſoſceles Triangles, the Aggregate of whoſe Baſes makes the Circumference, and their common Vertex is at the Center.

COROL. II.

Hence the Solidity of the Sphere will be gain'd by Multiplying its Surface by $\frac{1}{3}$ of its Radius.

Let the Square *a d*, the Quadrant *c b d*, and the right angled Triangle *a b d*, be ſuppos'd all three to to revolve round the Line *b d* as an Axis: Then will the Square generate a Cylinder, the Quadrant an Hemiſphere, and the Triangle a Cone, all of the ſame Baſe and Altitude.



I. Then the Square of *e h* (which is equal to the Square of *f d*, which is equal to the Square of *f h* together with that of *h d* (or its equal *g h*) will be equal to the Square of *g h* (= *h d*) together with the Square of *f h*. And ſince Circles are as the Squares of their Diameters (which muſt be now taken for granted, but will be proved in the 6th Book) the Circle made by the Revolution of *e h*, muſt be equal to the two Circles made by the Motion of *f b* and *h g*. Wherefore.

II. If you take the Circle made by the Revolution of *f h* from both, there will remain the Circle made by the Motion of *g h* equal to the Ring deſcribed by the Motion of *e f*: And thus, it muſt always be wherever you draw the Line *e h*, or *i m*, &c. III. There-

III. Therefore the Aggregate of all the Rings made by the Revolution of the *ef*'s, muſt be equal to *that* of all the Circles made by the Motion of the *gb*'s: (*i. e.*) the Diſh-like Solid, formed by the Revolving Rings, will be equal to the Cone formed by the Revolution of the *g b*'s, which are the Elements of the Triangle *a b d*. That is, the Diſh-like Solid will be, as the Cone is, $\frac{1}{3}$ of the Circumſcribing Cylinder, and conſequently the Hemiſphere muſt be $\frac{2}{3}$ of it: Wherefore the Sphere is $\frac{2}{3}$ of the Circumſcribing Cylinder.

IV. Let then the Radius of the Sphere be $r = cd = bd$, then the Diameter will be $2r$; let the Surface of the Sphere generated by the Revolving Semicircle be called S; and that of the Cylinder, formed by the Revolution of $2ac = 2r =$ Diameter, be called *ſ*. Wherefore in what was juſt now proved; (*Art.* by 33. of this Book) the Expreſſion for the Solidity of the Sphere in this Notation will be $\frac{rS}{3}$ – and putting c equal to the Circumference of the Baſe, or for the Periphery of a great Circle of the Sphere, the Curve Surface of the Cylinder (by Multiplying the Altitude into the Periphery of the Baſe) will be $2rc$; alſo $\frac{rc}{2}$ will be the Area of a great Circle (by Prop. 26 of Book 4.) and this Multiplied by $2r$, makes $\frac{2rrc}{2}$, which is the Solidity of the Cylinder (by *Cor. Art.* 11.) Now ſince *ſ* was put equal to $2rc =$ to the Curve Surface of the Cylinder: $\frac{Sr}{2}$ (by Subſtituting *ſ* for $2rc$) will be alſo = to the Solidity of the Cylinder. Now ſince the Sphere is = $\frac{2}{3}$ of the Cylinder; $\frac{rS}{3} =$

$=\frac{2}{3}\frac{rs}{2}$. That is $\frac{rS}{3}=\frac{2rS}{6}=\frac{rS}{3}$. Wherefore $rS=rs$; that is, dividing by r, $S=s$; or the Surface of the Sphere is equal to the Curve Surface the Cylinder: But the Curve Surface of the Cylinder was $2rc$;

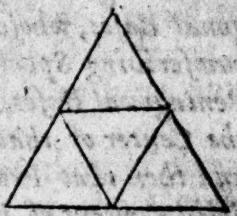
Wherefore to find the Area of the Surface of either Sphere or Cylinder, you muſt Multiply the Diame- ($=2r$) by the Circumference of a great Circle of the Sphere, or by the Periphery of the Baſe. From this Notation alſo $\frac{rc}{2}$, the Area of a great Circle of the Sphere, is plainly $\frac{1}{4}$ of $2rc$ the Surface of the Sphere. That is, the Surface of the Sphere is Quadruple of the Area of a great Circle of it.

V. Wherefore to $2rc$ the Convex Surface of the Cylinder, add rc equal to the Area of both its Baſes (each of which is $\frac{rc}{2}$) you will have $3rc$; which ſhews you that the Surface of the Cylinder (including its Baſes) being $3rc$, is to the Surface of the Sphere, which is, $2rc$, as three is to two: Or that the Sphere is $\frac{2}{3}$ of the Circumſcribing Cylinder, in Area, as well as Solidity.

34. Of all Solid Figures that can be encompaſs'd or determinated by the ſame Surface, the greateſt is a Spherical One, by *Art.* 13th of this Book, and *Art.* the laſt of Book the 4.

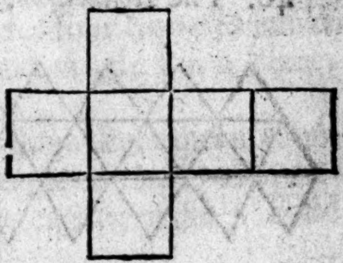
35. That is call'd a *Regular Body*, whoſe Surface is composed of Regular and Equal Figures. And whoſe Solid Angles are all equal, as are ——

36. The *Tetrahedron*, which is a Pyramid, comprehended under four equal and equilateral Triangles; ſo that its Baſe is equal to each Side.



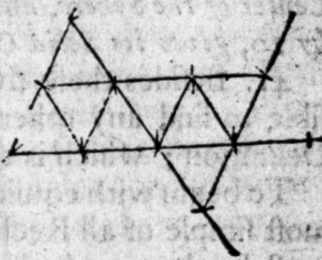
Wherefore its Solidity will be found by Multiplying the Baſe by $\frac{1}{3}$ *of the Altitude; which is the general way for all Pyramids.*

37. The *Hexahedron* or Cube, whoſe Surface is compos'd of ſix equal Squares, like Dice which are us'd in play.



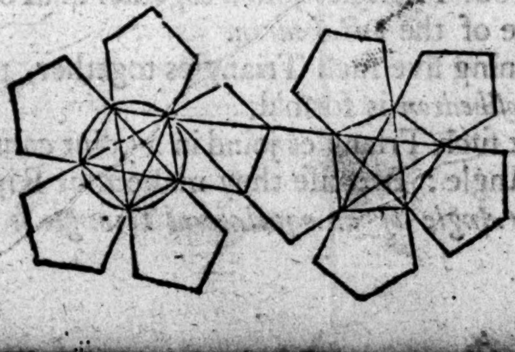
Its Solidity will be found by Cor. of Art. 12.

38. The *Octahedron*, which is bounded by eigh equal and equilateral Triangles.



This Figure is two Pyramids put together at their Baſes: Wherefore its Solidity is had by Multiplying the Quadrangular Baſe of either (here they are both join'd together in the middle of the Figure) by one third of the Perpendicular Altitude of one of the joined Pyramids, and then doubling the Product.

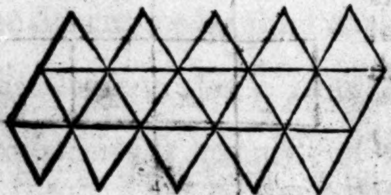
39. The *Dodecahedron*, which is contained under



twelve equal and equilateral Pentagons.

This Figure conſiſts of twelve Pyramids, with Pentagonal Baſes, whoſe common Vertex is the Center of a Circumſcribing Sphere: Wherefore any one of theſe twelve Pentagonal Baſes, Multiplied by $\frac{1}{3}$ of the diſtance between the Center of that Baſe, and the Center of the Sphere; and then that Product Multiplied by twelve, gives the Solid Content of this Regular Body.

40. The *Icoſihedron*, conſiſting of twenty equal and equilateral Triangles.



This Figure is compoſed of 20 Triangular Pyramids, all equal to one another, and whoſe Vertex is the Center of a Circumſcribing Sphere: Wherefore any one of the 20 Triangular Faces, Multiplied by $\frac{1}{3}$ of the Diſtance between the Center of the Face and the Center of the Sphere, and that Product Multiplied again by 20, gives its Solid Content.

41. Beſides theſe five *Regular Bodies*, 'tis not poſſible to find any others that ſhall correſpond to the Definition; which is thus Demonſtrated.

To begin with equilateral Triangles, which are the moſt ſimple of all Rectilineal Figures. Of theſe there muſt be three at the leaſt to make a Solid Angle, and three of them join'd together will juſt make the *Tetrahedron*. For thoſe three Triangles meeting in a point do form a Triangular Baſe ſimilar and equal to the Sides; as appears by the bare Compoſition of the Figure. Four Triangles join'd together in aPoint make the Angle of the *Octahedron*.

By joining five ſuch Triangles together, the Angle of the *Icoſihedron* is form'd.

But ſix ſuch Triangles joind in a point cannot make a Solid Angle: Becauſe they make four Right Ones (*for every Angle of an equilateral Triangle is $\frac{1}{3}$ of two,* *or* $\frac{2}{3}$ *of one Right Angle, either of which Fractions Multiplied by ſix gives four Right Angles.)* Whereas every Solid Angle is made up of ſuch plane Angles as all together muſt be leſs than four Right Ones (5. 25.) So that with Triangles 'tis impoſſible to form any more Regular Bodies than theſe three.

Next, if you take *Squares* and join three of them together, they will make the Angle of the Cube: And there can no other Regular Body but a Cube be made with Squares, for four Squares join'd together, will not make a Solid Angle but a Plane. (5. 25.)

If you join the Angles of three Pentagons together, you will conſtitute the Angle of the Dodecahehedren: But four ſuch Angles cannot make a Solid One.

And laſtly, *Three Hexagons* joined together do make juſt four Right Angles, and therefore they cannot make a Solid Angle: And as for three *Heptagons*, or other Figures of yet more Sides, they can much leſs do it; *(becauſe their Angles being very Obtuſe, three of them will exceed four Right Ones.)* So that upon the whole 'tis plain, that of theſe five *Regular Bodies*, three are made of Triangles, one of Squares, and one of Pentagons, *and there can be no other.*

ELEMENTS
OF
GEOMETRY.

BOOK VI.

Of Proportion.

1. WHEN we ſpeak of *Magnitude* ; and ſay, that any Quantity is great, we always make a Compariſon between that Quantity and ſome other of the ſame Nature, in reſpect to which we ſay that it is Great.

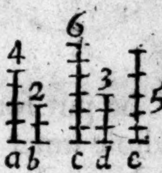
Thus we ſay of an *Hill*, that 'tis Little ; or of a *Diamond*, that 'tis Large ; becauſe we compare that Hill, with others that are Higher, and in reſpect of them 'tis Little ; and we compare that Diamond, with other that are Little, and in reſpect of them, we ſay 'tis a Large one.

2. When we conſider one Quantity in reſpect of another, to ſee what Magnitude it hath in compari-

ſon of that other: The Magnitude ſo found, is call'd its *Ratio* or *Reaſon*; tho' it would be more intelligible if it were call'd Compariſon.

3. That Quantity which is compar'd with another is call'd the Anticedent; and *that other* with which it is compar'd, is call'd the Conſequent.

4. When we conſider four Quantities, and compare them (by Pairs) two with two; as *a* four, with *b* 2, and *c* 6, with *d* 3. If we find that *a* hath as much Magnitude (or is as big) in compariſon of *b*, as *c* hath in compariſon of *d*; (i. e. *When we find that* e *is contained in, or doth contain* b : : *as often as* c *is contained in, or doth contain* d) Then we ſay, that their Ratio's are equal; that is, the Ratio that *d* hath to *b*, is equal to the Ratio of *c* to *d*: For as *a* is twice as big as *b*; ſo *c* is twice as big as *d*.



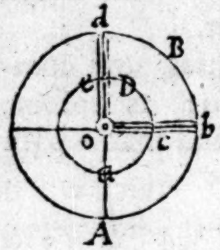
5. But if *a* 4, hath more Magnitude in reſpect of *b* 2, than *c* 6 hath in reſpect of *e* 5. That is, if as *a* 4 is twice as big as *b* 2, *c* 6 be found not to be twice as big as *e* 5. Then the Ratio's are unequal: And we ſay, *a* hath a greater Ratio to *b*, than *c* hath to *e*. So that to have a greater Ratio, is nothing but to have more Magnitude, or to be bigger, in reſpect of a ſecond Term, than a third is in reſpect of a fourth.

6. The Equality of Ratio's is call'd Proportion; and when we find that of four Quantities or Numbers, the firſt hath as much Magnitude *(or is as big)* in reſpect of the ſecond; as the third is in reſpect of the fourth; then we ſay, that thoſe four Quantities are *Proportionals*.

The better to make the Myſteries of Proportion comprehended, which paſs for the moſt difficult things in Geometry, *as unqueſtionably they are moſt Important, I will explain them by an Example; which (in my Opinion)*

nion) will render all theſe things very **Intelligible,** *which otherwiſe appear very perplext.*

7. Let us imagine the Circle *b* A *d* to be deſcrib'd by the Motion of the Line *o b*, round the Center *o*. And at the ſame time, let the Circle *c a e* be deſcribed by the Motion of a Point *c*, in the Line *o b*. Let us ſuppoſe alſo that the Line *o b* be moved once round again, and at laſt to ſtand in the Poſition *o d*. Let the Ark *d* B *b* be called B, and the Ark *e* D *c*, be call'd D. Let A be put for the whole outer Circle, and *a* for the whole inner one.



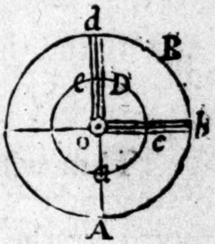
Now if we compare the whole Circle A with its Ark B, and the whole other Circle *a* with its Ark D. We ſhall find plainly, that the Circle A is juſt as big in reſpect of the Ark B, as the inner one *a* is in reſpect of the Ark D; and therefore if B be a fourth, or any other part of the Circle A, D alſo will be a fourth, or the ſame proportional part of its Circle *a*. Which we uſually expreſs by ſaying, as A is to B, ſo is *a* to D. And write it thus A. B : : *a*. D. or 24. 6 : : 8. 2.

8. If you ſhould change the Order of the Terms, and compare B with A and D with *a*; you will find plainly that B. A : : D. *a*. So that ſuppoſing A. B : : *a*. D. we cannot but preſently conclude by Inverſe Proportion, that B. A : : D. *a*.

9. If you change them ſo as to compare Antecedent with Antecedent, and Conſequent with Conſequent; you will find Alternately, that A. *a*. : : B. D. And this is very plain; for if the whole Circle A be double, triple of, or in any other Proportion, to the Circle *a*, the Ark B muſt be alſo double, triple of, or in the ſame Proportion to the Ark D; for Aliquot Parts will be as their wholes. This I ſay is

plain, becauſe the two Circles A and *a* are deſcrib'd by the Motion of the Line *o c b*; ſo that while *b* deſcribes the Circle A, *c* deſcribes the inner Circle *a*; and while *b* deſcribes the Ark B; *c* alſo deſcribes the Ark D. And this by one common circular Motion; only the Point *c* moving much ſlower than the point *b*, deſcribes a Circle much leſs, in proportion to the ſlowneſs of its Motion: Thus alſo when the Point *b* ſhall have deſcribed the Ark B, the point *c* in like manner will have deſcrib'd the Ark D, which will be much leſs than B; in proportion to the ſlowneſs of its Motion: in Numbers 24. 8 :: 6. 2.

10. If we compare the differences between the Antecedents and Conſequents, with their Conſequents; as for Inſtance, A leſs B with B, and *a* leſs D with D, we ſhall find they alſo are Proportional: And that A leſs B B :: *a* leſs D. D. 18. 6 :: 6. 2.



For 'tis manifeſt that the Ark *b* A *d* (which is A leſs B) is to B, as the Ark *c a e* (which is *a* leſs D) is to D. And this is call'd *Proportion by Diviſion.*

11. If we add the Antecedents and Conſequents together; we ſhall find that A, more B, is to B :: as *a*, more D is to D. Which is call'd *Compoſition.* In Numbers 30. 6 :: 10. 2.

12. And if we ſhould ſay, that A. A leſs B :: *a*. *a* leſs D. This kind of Proportion is call'd *Converſion.* You may alſo infer by way of *Mixing* the Terms, as ſome call it. That A + B. A — B :: *a* + D. *a* — D: or that A + B. *a* + D :: A — B. *a* — D. &c. That is, 30. 18 :: 10. 6 and 30. 10 :: 18. 6, &c.

And it will be very convenient for the Learner to enure himſelf to all the Changes and Varieties of Proportion, and to have them ready in his Mind; Becauſe a great many Propoſitions in Geometry, as they have been delivered by the Ancients, and purſued by the Moderns that have trod in their ſteps, are demonſtrated by Compoſition, Diviſion, Alternation, and Intermixing of Proportion.

13. If never ſo many Quantities, are thus Proportional: It will be as any one Antecedent to its Conſequent :: So is the Sum of all the Antecedents to the Sum of all the Conſequents. *v. gr.*

If 4. 12 :: 2. 6 :: 3. 9 :: 5. 15 : then ſhall
14. 42 :: 4. 12.

14. If *a*. *b* :: *c*. *d*.
4. 12 :: 3. 9 and alſo

b. *f* :: *d*. *g*.
12. 36 :: 9. 27

Then will it be by Proportion of Equality.

a. *f* :: *c*. *g*.
4. 36 :: 3. 27.

The Reaſon of which is plain, if you conſider, That ſince *b*. *f* :: *d*. *g* : *f*, and *g*, muſt needs be either *Similar aliquot Parts*, or *Equimultiples* of *b* and *d*. And therefore ſince *a* and *c*, are to *b* and *d*, in the ſame Ratio as *b* and *d* are to *f* and *g*, *a* muſt alio be in the ſame Ratio to *f*. the *Part* or *Multiple* of *b* :: as *c* is to *g*. the *Part* or *Multiple* of *d*.

If *a*. *b* :: *c*. *d*
12. 4 :: 9. 3 and then,

b. *f*. *h*. *c*.
4. 2. :: 18. 9

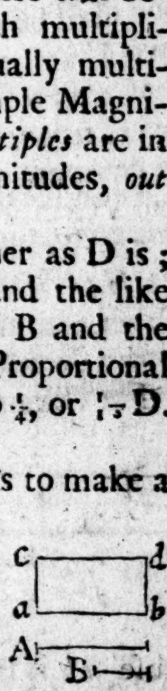
Then will *a*. *f* : : *b*. *d*
12. 2 : : 18. 3.

Which is called, *Proportio ex equo perturbata :* and this muſt be true : becauſe 12 containing 4 as oft as 9 contains 3 : and 4 containing 2, as oft as 18 contains 9 ; 12 muſt contain 2. as often as 18 contains 3. Wherefore this is only the orderly Proportion of Equality diſturbed, and therefore is by ſome called *Inordinate Proportion.*

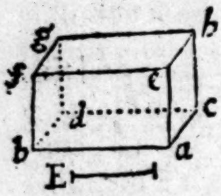
15. If B be taken as often as D. *ex gr.* 3 B and 3 D. we may conclude that B. D : : 3 B. 3 D. or as 10 B to 10 D. alſo as 12 $\frac{1}{2}$ B. to 12 $\frac{1}{2}$ D. And ſo on in whatſoever Proportion the two Magnitudes B and D are multiplied, ſo they are multiplied equally, *or that you take one as often as you take the other.* For then there will be the ſame Proportion between the Magnitudes thus equally multiplied, as there was between the ſimple Magnitudes, before ſuch multiplication. And theſe Magnitudes thus equally multiplied are call'd *Equimultiples* of the ſimple Magnitudes B and D ; and we ſay that *Equimultiples* are in the ſame Proportion as ſuch ſimple Magnitudes, *out of which they are compounded.*

16. If B be divided in the ſame manner as D is ; and *ex gr.* you take a fourth part of B and the like of D. or the tenth or any other part of B and the ſame of D. Then will theſe Parts be Proportional to their wholes, B. D : : $\frac{1}{4}$ B (or $\frac{1}{10}$ B.) is to $\frac{1}{4}$, or $\frac{1}{10}$ D. All which is Self-evident.

17. To multiply one Line by another is to make a Rectangl'd Parallelogram, whoſe two Contiguous Sides ſhall be the two Lines given. Thus, if you multiply the Line A by B ; 'tis the ſame thing as to make the Rectangle *a b c d* ; whoſe Side *a b* is equal to A, and *a c* to B.



18. To multiply a Rectangle, or any other Surface by a Right line, is to make a Rectangled Parallelopiped *(or Priſm)* (5. 9) whoſe Baſe ſhall be the Surface given, and its Perpendicular height the Line given.

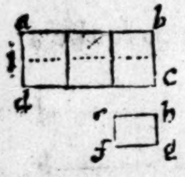


Thus to multiply the Surface *a b d c* by the Line E, is the ſame thing as to make a Solid *a b f g h e* whoſe Baſe is the Surface given *a d*, and its height *a e* or *b f*, equal to the Line given E.

19. All Magnitudes may be expreſs'd by Lines: As if one Magnitude be double or triple of another, or in any other *Ratio*, two Lines may eaſily be taken, of which one ſhall be double or triple of the other, or in any other like Proportion with thoſe Magnitudes: So for Inſtance, to expreſs two times, as one Hour and two Hours; or two Velocities, of which one ſhall be double to the other; you need only take two Lines, as *a* double of *b*; and then you may ſay that *a* repreſents two Hours or Velocities, and *b* anſwers to one of each: and then you may proceed to compute with thoſe two Lines, as with the Hours and Velocities themſelves, &c.

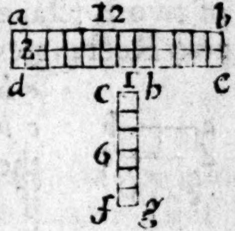
20. To know the Proportion of Rectangles, the *Ratio* of the Length of One to the Length of the Other, and moreover the *Ratio* of the Breadth of One to the Breadth of the Other muſt be known.

For Example; To know what Proportion the Rectangle *a c* hath to *e g*; 'Tis not enough only to know that the Length *a b* is triple of *e h*; but it muſt be known alſo, that *a d* is double of *e f*. For if *a i* be taken equal to *e f*, the Rectangle *b i* will be triple of *e g*, becauſe *a b* is triple of *e h*, and *a i* equal to *e f*. And moreover becauſe *i d* is alſo equal to a *i*, or *e f* (for

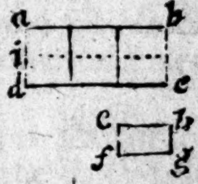


a d is ſuppoſed to be double of *a i*, and of *e f*) the Rectangle *i c* ſhall alſo be triple of *e g*; ſo that the whole Rectangle *a c* is twice triple of the Rectangle *e g*; that is, ſextuple of it, or containing it ſix times. And what we ſay now only of the double or triple *Ratio* of their Breadths and Lengths, is alſo to be underſtood of any other *Ratio*, be it what it will: For if *a b* be quadruple of *e h*, and *a d* triple of *e f*, the Rectangle *a c*, will be three times quadruple of the Rectangle *e g*: that is duodecuple of it, or doth contain it twelve times.

But if *a b* be duodecuple of *e h*, and at the ſame time *e f* be Triple of *a d*, then there is a certain Compenſation made: For if Reſpect were had to their Breadths *a b* and *e h* only, the Rectangle *a c* would exceed the other, nay indeed contain it 12 times: Nevertheleſs this exceſs is loſt (*in ſome Meaſure*) in reſpect of their Altitudes or Heights *a d* and *e f*, which if only conſider'd, the Rectangle *e g* would be Triple of *a c*. But then when we come to compare theſe ſeveral Exceſſes and Deficiencies together; we ſhall find that the Rectangle *a c* being one way 12 times greater, and the other way three times leſs than *e g*, will be at laſt but only four times as great.



21. And this is what we mean, when we ſay, that all Rectangles are to each other in a *Ratio Compounded* of that of their ſides; for if *a b* be triple of *e h*, and *a d* double of *e f*, the Rectangle *a c*, ſhall be to the Rectangle *e g* in a *Ratio* compounded of the triple and the double, that is, it ſhall be thrice double, or twice triple, or in one Word Sextuple. So alſo if *a b* were quadruple of *e h*, and *a d*

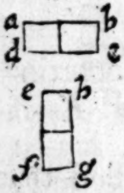


triple of ef; the Rectangle ac would then be to eg in a *Ratio* compounded of the Quadruple and the Triple; ſo that it would have been three times Quadruple, or four times Triple, or in one word Duodecuple of eg.

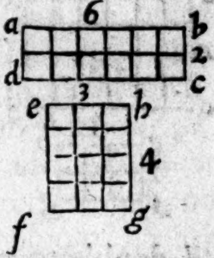
Moreover, if ab were Duodecuple of eh, and ad Subtriple of ef, (that is, if ef be Triple of ad) the *Ratio* of the Rectangle ac to eg would be compounded of the duodecuple and ſubtriple *Ratio*; ſo that ac, would have been 12 times ſubtriple of, or in one word Quadruple of eg.

If you take the third part of a Crown 12 *times it will make, or be equal to four whole Crowns: So that four Crowns are* 12 *times ſubtriple of one Crown; that is, do make* 12 *thirds of a Crown.*

22. From whence it will appear that if the ſides of two Rectangles are reciprocally Proportional, thoſe two Rectangles are equal: For if ab be double to eh, and reciprocally hg be double to cb: Or if ab be triple of eh, and then hg be triple of bc; or in a word, if whatever *Ratio* ab hath to eh, hg hath back again the ſame *Ratio* to bc, 'tis plain that as much as the firſt Rectangle ac exceeds the other in length, juſt ſo much is it exceeded by the other in breadth; ſo that the length of one Compenſates for the breadth of the other, and conſequently they muſt be equal. And from hence is deduced this moſt uſeful and important Propoſition: That,



23. If four Quantities (*or Numbers*) be proportional, the Product ariſing from the Multiplication of the two middle Terms, is always equal to that, which is made by the Multiplication of the two Extreams. As if *a b. e h* : : *h g, b c.*



I ſay, from the Multiplication of the extreams *a b* by *b c* there is produced the Rectangle *a c*: And by multiplying the middle Terms *e h* and *h g*, there is produced the Rectangle *e g*; and thoſe two Rectangles *a c* and *e g* are equal. (6. 27). (*Becauſe, as much longer as* a b *is than* e h, *juſt ſo much longer is* h g *than* b c), On which is founded the Reaſon of the Golden Rule.

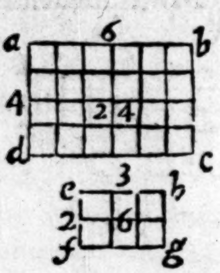
COROLLARY.

Hence, if in two ranks of *Diſcrete* Proportionals, the four middle Terms are the ſame: As if *a. b* : : *c, d*, and then alſo *e. b* : : *c. f*. I ſay it will be as *a. e* : : So will reciprocally *f*, be to *d*: For ſince the middle Terms are the ſame in both, the Rectangle *a d* will be equal to *e f*, and conſequently their Sides muſt be reciprocally proportional; that is *a, e* : : *f. d.*

What is thus done by Lines and Rectangles, may be done by any Quantity whatſoever; becauſe all Quantities can be expreſs'd by Lines, and all Multiplications of Magnitudes by Multiplications of Lines, *i. e.* by Rectangles. (6. 24).

24. When Rectangles have their ſides Proportional, ſo that *a b. e h* : : *a d. e f.* then is the Rectangle *a c* to the Rectangle *e g*, in a *Duplicate Ratio*, to that of their Sides: For the *Ratio* of *a c* to *e g*, is compoun-

ded of the *Ratio* of *a b* to *e h*, and of the *Ratio* of *a d* to *e f*. (6. 26). But the *Ratio* of *a b* to *e h* is in this Caſe, (by the Suppoſition) the ſame as the *Ratio* of *a d* to *e f*; ſo that to gain the *Ratio* which the Rectangle *a c* hath to *e g*, we need only take *twice the Ratio* of *a b* to *e h*. For Example, if as here *a b* be double to *e h*, and *a d* double to *e f*, the Rectangle *a c* ſhall be twice double, that is, Quadruple of the Rectangle *e g*. And if *a b* had been triple of *e h*, and conſequently *a d* triple of *e f*: Then the Rectangle *a c* would have been three times triple, that is nine times as big as *e g*; Or if *a b* had been Quadruple of *e h*, *a c* would have been 16 times as great as *e g*.

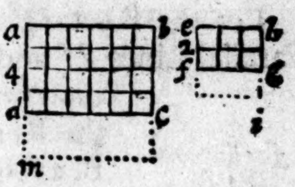


25. If a third Line be taken as *n o*; and it be ſo proportional that *a b*. *e h* : : *e h*. *n o*. Then ſhall the two Rectangles *a c* and *e g* be to one another as the two Lines *a b* and *n o* (vid. *Fig. Preced.*)

For *a b* is to *n o* in a duplicate *Ratio* of *a b* to *e h*. And if *a b* had been, (as it is double,) triple or quadruple of *e h*: Then would *a b* have been in a *Ratio* three times Triple; or four times quadruple of (*that is nine or* 16 *times as great as*) the *third Proportional* n o.

26. Thoſe Rectangles which have their ſides thus Proportional: That *a b*. *e h* : : *a d*. *e f*, are called *Similar*, whoſe *Homologous Sides* are thoſe which anſwer each to other in the Proportion, as *a b* and *e h*, or *a d* and *e f*: For as *a b* is the greateſt ſide of the Rectangle *a c*, ſo *e b* is alſo the greateſt ſide of the Rectangle *e g*.

27. All Squares are ſimilar Rectangles. For 'tis plain that if *a b* be double or triple of *e h*, *a m* muſt alſo be double or triple of *h i*: Becauſe *a m* is equal to *a b*, and *h i* to *e h*.



28. All ſimilar Rectangles are to each other as the Squares of their Homologous Sides. I ſay the Rectangle *a c* is to the Rectangle *e g* :: as the Square *b m* to the Square *ei*: For as well Squares as Rectangles are to one another in a Duplicate *Ratio* of *a b* to *e h* (6. 29, 30).

29. To know the *Ratio* between two Solid Rectangles, or *Parallelopipeds* there ought to be known the ſeveral *Ratio's* that their Baſes and Heights have to each other; becauſe the *Ratio* of one Solid to another is compounded of the *Ratio's* of their Lengths and Breadths and Thickneſſes or Heights; as is eaſie to concieve,if that be well underſtood which hath been ſaid about the Proportions of Rectangles. For if one Parallelopiped hath its Baſe double to the Baſe of another, and its Height, triple of the Height of the other: The former will be twice triple, or three times double, or in one word Sextuple of the latter.

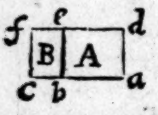
30. If the *Baſes* of two Parallelopipeds be *Reciprocally* as theit *Heights*, thoſe Parallelopiped are equal: Which is proved by the 27th of this Book; for as much as one exceeds the other in Breadth andLength, ſo much doth the other exceed it in height.

31. When Parallelopipeds have all their ſides Proportional, they are called *Similar*; and they are in a *Triplicate Ratio* of their Sides,as it hath been proved of Rectangles, that they are in a *Duplicate Ratio* of their Sides

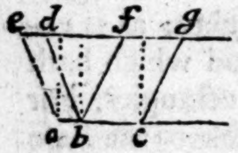
32. Similiar Parallelopipeds are to one another as the Cubes of their Homologous Sides; for both Cubes and Parallelopipeds are in a *Triplicate Ratio* of their Homologous Sides.

34. All Rectangles having the ſame or equal Heights, are to one another as their Baſes, and having the ſame Baſſes their Heights are equal.

Let the Rectangles A and B be between the ſame Parallel Lines *df* and *ca*; ſo that *ad* be equal to *cf*: then do I ſay, that A. B : : *a b. b c* That the Rectangle A. is to the Rectangle B. as the Baſe *a b.* to the Baſe *b c*: And that if, for Inſtance, *a b* be double to *b c*, then ſhall A be double to B. For A is nothing but the Line *b a* multiplyed by *d a.* (6. 17); and B is nothing but the Line *c b* multiplied by the ſame Line *a d* or (which is all one) *b e* or *f c*. Wherefore (6. 15) A. B : : *a b. b c.*



35. All *Parallelograms* which are between the ſame Parallels (*or which have the ſame Height*) are as their Baſes. I ſay the Parallelogram *e b* is to the Parallelogram *b g* : : as the Baſe *a b* is to the Baſe *b c*. For having made the two Prick'd Rectangles on the ſame Baſes, thoſe will be equal to the Parallelograms, (by 3. 14). But thoſe Rectangles are as their Baſes (by the Precedent). Wherefore the Parallelograms muſt alſo be as their Baſes; That is *e b. b g* : : *a b. b c.*



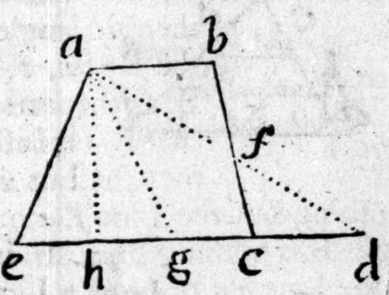
36. All Triangles (*which have the ſame Heights*) or are between the ſame Parallels, are as their Baſes; For they are halves of Parallelograms. (3. 8).

37. When Triangles (as thoſe in the following Figure) have their Baſes on one and the ſame Line, and their *Vertices* or tops meeting in the ſame Point: they are taken to be between the ſame Parallels, as *a d e* and *c d e*, and *a d e* and *b d e* (*becauſe they have the ſame Perpendicular Height.*)

PROB.

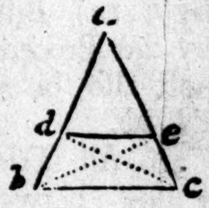
PROBLEM I.

Hence may a Trapezium as *a b c e*, whoſe two ſides *a b* and *e c* are parallel, be divided in any given *Ratio*.



For take *c d* = *a b* and draw *a d*, then will the △s *a b f* and *f c d* be equal (by 14. 2) and conſequently the △ *e a d* = Trapez. *e a b c*. Wherefore if you divide *e d* the Baſe of the △ *e a d* into any number of parts or according to any *Ratio*, Lines drawn from the Vertex to ſuch Diviſions of the Baſe will divide the △ *e a d*, and conſequently the Trapezium, in the ſame *Ratio*.

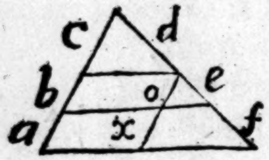
38. If in any Triangle a Line be drawn Parallel to the Baſe, that Line ſhall cut the Legs Proportionally. Let the Triangle be *a b c*, and let the Line *d e* be Parallel to *b c*.



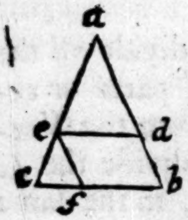
I ſay that *a d*. *a e* : : *a b*. *a c* : : *d b*. *e c*, &c. Draw the Lines *d c* and *e b*, then ſhall the Triangle *e c d* be to *e a d*, as the Baſe *e c*, is to *a e*. (6. 40, 46). So alſo the Triangle *d e b* is to *e a d* : : as the Baſe *d b*. is to *d a*. But the Triangle *e c d* is equal to *d e b* (3. 15); wherefore the Triangle *b d e* (or *c e d*) is to the Triangle *e a d* : : as *b d*. is to *d a* : : or as *c e* to *e a*. Therefore alſo muſt *b d*. *d a* : : *c e*. *e a*, becauſe both the *Ratio* of *b d* to *d a*, and alſo that of *e c*. *e a*, are the very ſame with that of the Triangle *b e d* or *c e d*, to the Triangle *a d e*.

COROL-

COROLLARY.



If many Lines are drawn parallel to the Baſe of any Triangle the Segments of the Sides a, b, c. and d, e, f will be proportional, for drawing $o\,x$ parallel to $a\,b\,c : b = o$ and $a = x$ but $x . o : : f . e$: wherefore $a . b : : f . e$. Q. E. D.

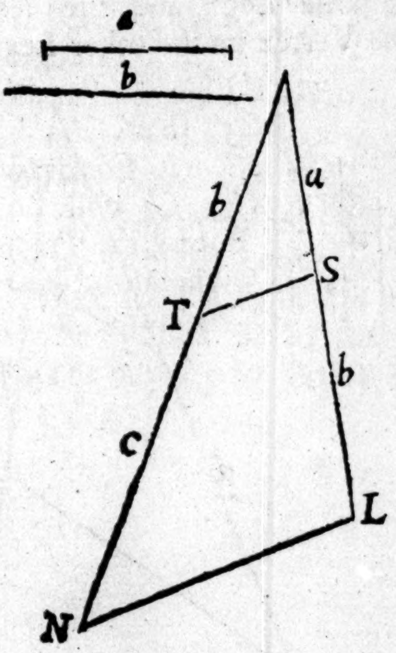


39. If in a Triangle, as $a\,c\,b$, you draw a Line $d\,e$ Parallel to the Baſe $c\,b$, I ſay, that $e\,d . c\,b : : a\,e . a\,c : :$ or as $a\,d . a\,b$. For drawing $e\,f$ Parallel to $a\,b$; $f\,b$ will be equal to $e\,d$ (3. 9). But by the Precedent $f\,b . c\,b : : a\,e . a\,c$. wherefore $e\,d .$ ($f\,b$) $c\,b : : a\,e . a\,c$. or as $a\,d$. to $a\,b$.

PROBLEM I.

Two Lines a *and* b *being given, to find* c, *a third Proportional to them.*

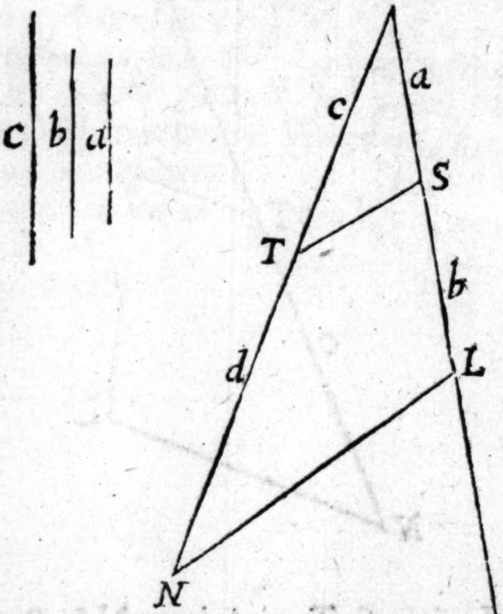
Make any Rectilineal Angle, and from the Vertex or Top of it, ſet the two given Lines down on the Legs, as you ſee in the Figure. Set alſo *b* downward from S to L, join S T, and draw N L Parallel to it ſo ſhall T N be *c*, the Line ſought; For *a*. *b* :: *b*. *c*. by this Propoſition.



PROBLEM II.

Iſ three Lines as a, b *and* c *had been given, to find a fourth Proportional (as* d*) to them, or to work the Rule of Three in Lines, you muſt proceed thus.*

Set the two firſt Lines *a* and *b* from the Vertex down on the ſame Leg; and then ſet *c* the third Line, from the Vertex on the other Leg: Draw the Line T S, and thro' the Point L draw L N Parallel to it; So ſhall T N be equal to *d* the fourth Proportional ſought; for : *a. b* : : *c. d.* by the Precedent Propoſitions.



PROB.

PROBLEM III.

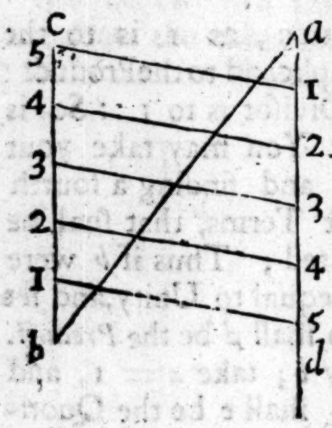
And this way 'tis very eaſie to find a Line that ſhall expreſs the Product *of any two Numbers or Quantities: Or the* Quotient *of one divided by the other.*

For ſince in all Multiplication, as 1: is to the Multiplicator :: So is the Multiplicand to the Product: And ſince in Diviſion as the Diviſor is to 1 :: So is the Dividend to the Quotient: You may take your 1. of any Length off a Scale, and finding a fourth Proportional to the three firſt Terms, that ſhall be a *Product*, or a *Quotient* required; Thus if *b* were to be multiplied by *c*: make *a*, equal to Unity, and ſet off *b* and *c* as before ſhew'd, ſo ſhall *d* be the *Product*. Or if *d* were to be divided by *b*; take $a = 1$, and ſet off all things as before; ſo ſhall *c* be the Quotient; for $b . a :: d . c$.

PROBLEM IV.

To divide a given Line a b *into any Number of equal Parts: As ſuppoſe into Six.*

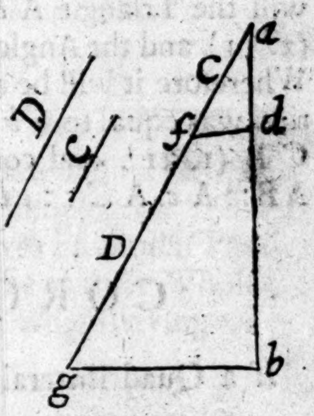
Make at *a* and *b* any two equal Angles, and on the Legs *d a* and *e b* run with a pair of Compaſſes, five equal Diviſions (for they muſt be always one leſs in Number than the required Diviſion or Parts of the Line given) drawing alſo Lines acroſs from one Point to the other, as you ſee in the Figure; ſo ſhall thoſe Lines divide the given Line *a b* into the ſix Parts required: For the croſſing Lines being Parallel one to another, muſt divide *a b* in the ſame Proportion as *a d* and *b e* are divided.



PROBLEM V.

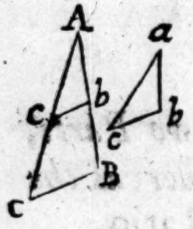
To divide a given Line ab *into two Parts; ſo that they ſhall be to each other as the Line* C *to* D *; or in any given* Ratio.

Make any Angle with the given Line *a b*, and ſet the Line *c* from its Vertex *a*, to *f*. And ſet the Line D from *f* to *g*; draw the Line *g b*, and thro' *f*, a Parallel to it, as *f d*: So ſhall the Point *d* divide *a b* in the Ratio required: For C. D : : *a d*. *d b*.



And much the ſame way may you cut off from any given Line *a b* any part or parts required; as ſuppoſe $\frac{2}{5}$.

Make any Angle as *g a b* as before, and ſet on the Leg *a g*, *a g* equal to five parts taken off from any Scale: Then ſet two ſuch parts from *a* to *f*, join *g b*, and draw *f d* Parallel to it; ſo ſhall *a d* be equal to $\frac{2}{5}$ of *a b*.

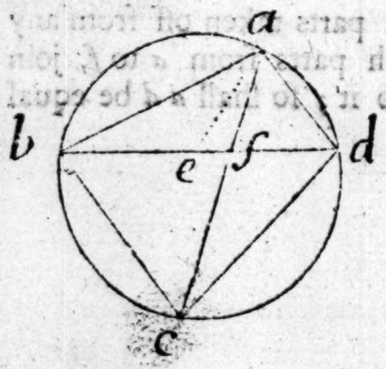


38. Thoſe Triangles are called *Like* or *Similar*, which have all their three Angles reſpectively equal to one another, *or which are Equiangular*: v. gr. If the Angle A be equal to *a*, the Angle B to *b*, and C to *c*, then the whole Triangle A B C is *Like* or *Similar* to the Triangle *a b c*.

39. All ſimilar Triangles have their Sides about the equal Angles Proportional. I ſay, A B. *a b* : : A C. *a c* : : B C. *b c*, &c For take, in the greater Triangle A B C, A *b* equal to *a b*, and *A c* equal to *a c*; then will the Triangle A *b c* be every way equal to *a b c* (2. 11). and the Angle A *b c* is equal to the Angle *a b c*: Wherefore it will be alſo to B, which by the ſuppoſition was equal to *a b c*, and therefore *c b* is Parallel to C B (1. 31). and conſequently (by 6. 42, 43), A *b*. A B : : A *c*. A C : : *c b*. C B.

COROLLARY I.

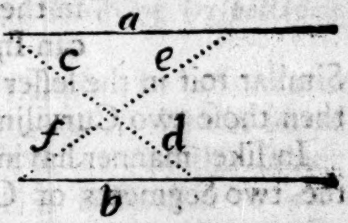
If a Quadrilateral Figure as *a b c d* be inſcribed in a Circle, the Rectangle under the Diagonals *b d* and *a c*, is equal to both the Rectangles under the oppoſite Sides: That is, $ac \times bd = \overline{ba \times cd} + \overline{ad \times bc}$, make the Angle *bae* = *c a d*; and then adding the Angle *e a f* to both, the Angle *b a f* = *e a d*: And the △ *a a d* ſimilar to △ *b a f*: Then will *a c*. *c d* : : *b a*. *b e*. (by this Prop.) Wherefore *a c* × *b e* = *c d* × *b a*. Again alſo



ſo ad. de : : ac, cb. Wherefore $ad \times cb = de \times ac$. But $ac \times be + ac \times ed = ac \times bd$. Wherefore $ac \times bd = ba \times dc + da \times bc$. Q. E. D.

COROL. II.

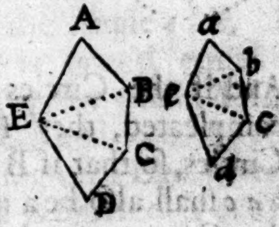
The Segments of Lines interſecting each other between two Parallels, are Proportional:



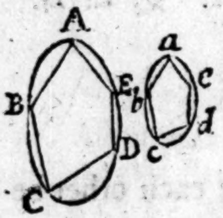
That is, c. d : : e. f. for by ſimilar Triangles c. e : : d. f. wherefore alternately c. d : : e. f. wherefore $cf = de$.

40. All ſimilar Triangles are in a Duplicate *Ratio* of, or as the Squares of their Homologous Sides; for ſimilar Triangles are the Halves of ſimilar Parallelograms, wherefore they muſt be as their Wholes.

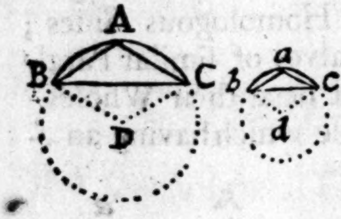
41. Similar Polygons are thoſe which having an equal Number of Sides have all the ſeveral Angles in one, equal to thoſe in the other, and alſo the ſides about thoſe equal Angles Proportional. As if the Angle A be equal to a, B to b; and moreover A B. ab : : B C. bc : : C D. cd. than thoſe two Polygons are Similar.



42. And among *Curvilineal* or *Mixt Figures*, thoſe are *Similar* in which you may inſcribe, or about which you may Circumſcribe ſimilar Polygons; ſo that any Polygon being Inſcribed or Circumſcribed about one Figure, you may Inſcribe or Circumſcribe a Similar one about the other. For inſtance, if having Inſcribed any Polygon as A B C D E in the greater *Curvilineal* Figure, you can Inſcribe another in all reſpects Similar toit in the leſſer Curvilineal Figure *a b c d e*, then thoſe two Curvilineal Figures are Similar.



In like manner having taken two mixt Figures as the two Segments of Circles B A C and *b a c*; and having Inſcribid in one any Triangle at Pleaſure as B A C, if then you can Inſcribe in the other Segment another Triangle *b a c*, that ſhall be Similar to the former; then ſhall thoſe two *Segments* be ſimilar Figures. And if the Circles of which they are Segments be compleated, they ſhall be ſimilar Parts of thoſe two Circles, ſo that if B A C be a third part of its Circle, *b a c* ſhall alſo be a third part of it Circle: And if to the Centers you draw the Lines B D and C D, and alſo *b d* and *c d*; the Angles D and *d* ſhall be equal. (See 4. 11. and the following Propoſitions.)

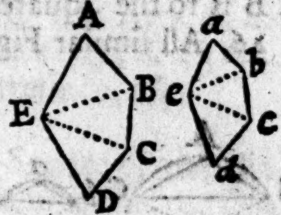


COROLLARY.

The Peripherys of Circles are as their Diameters: For as B A. *b a* : : B D. *b d* : : 2 B D. 2 *b d* : : ſo it will be of every ſide of the Inſcribed or Circumſcribed Polygon; wherefore the Sum of them all, that is the Peripherys, muſt be in the ſame *Ratio*.

43. All Circles are ſimilar Figures.

44. All ſimilar Polygons may be divided into an equal Number of ſimilar Triangles. Let the ſimilar Polgyons be A B C D E and *a b c d e*; and let the firſt be divided into Triangles by the Lines E B and E C (3. 14). I ſay that if the other be alſo divided into Triangles by the Lines *e b* and *e c*, all the Triangles in one ſhall be (*reſpectively*) Similar to thoſe in the other.

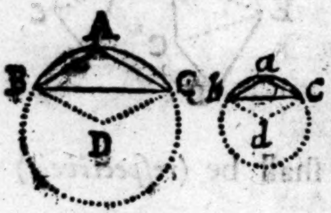


For Inſtance, I ſay the Triangle *a b e* is Similar to A B E: for the Angle *a* is equal to A by the ſuppoſition) and alſo A B. *a b* :: A E. *a e* (by the ſame); wherefore the Triangle A B E is Similar to *a b e*, (6. 46). Again, the Angle E B C may be proved equal to *e b c*; becauſe the Angle A B C is (by the Suppoſition) equal to *a b c*, and it was proved (*in the laſt ſtep, where the Triangle* A B E *was proved Similar to* a b e) that the Angle *a b e* is equal to A B E; wherefore from equal things taking away equal, the Angle E B C remains equal to the Angle *e b c*. In like manner the Angle *e c b*, is prov'd equal to E C B, and conſequently (6. 45) the whole Triangle *e b c* will be Similar to E B C; and ſo of the reſt.

Hence the Practice of making on a Line given a Polygon Similar to one aſſigned is derived. For dividing the given Polygon into Triangles, make a Figure conſiſting of a like number of ſimilar Triangles, on the given Line.

45. All ſimilar Polygons are to one another in a Duplicate *Ratio* of, or as the Squares of their Homologous Sides. I ſay as the Square of A B is to the Square of *a b* :: ſo is the whole Polygon A B C D E to the Polygon *a b c d e*. For ſince all the Triangles in one Polygon, are Similar to thoſe

in the other (6. 51). All in one Polygon will be to all thoſe in the other in a Duplicate *Ratio* of any of their Homologous Sides; that is, as the Square of A B is to the Square of *a b*.



46. All ſimilar Figures, even Curvilineal ones, are to one another as the Squares of any ſide of any ſimilar Figures, which can be Inſcribed or Circumſcribed about them *v. gr.* Let there be two Circles, in which are Inſcrib'd two ſimilar Triangles *a b c* and A B C: I ſay, the whole Circle A B C, is to the Circle *a b c* :: ſo is the Square of B C, to the Square of *b c*, or which is the ſame thing, as the Square of the Radius B D to the Square of the Radius *b d*. For in or about the Circle *a b c* may be Inſcribed or Circumſcribed any Polygon you pleaſe; (or at leaſt ſuch an one may be imagin'd) (4. 30). But every Polygon inſcrib'd in *a b c* will have a leſs *Ratio* to the Circle A B C, than the Square of *b c* hath to the Square of B C; and every one Circumſcribed about *a b c* will have a greater *Ratio* to the Circle A B C, as is eaſie to prove by the Precedent, and from what hath been ſaid of Circles in the fourth Book. Wherefore all ſimilar Figures, &c.

COROL. I.

I. Circles are to each other as the Squares of their *Radii* or Diameters: for ſuppoſe a Circle whoſe Radius is *r*, and then another Circle greater (or leſs) than that; and call its Radius R, then will its Diameter be 2 R: then whatever the *Ratio* of the Diameter (2 R) be to the Periphery, let it be expreſſed by the Letter e, then will $2Re$ (or e times the Diameter) be the Periphery ; and half of this, *viz.* Re, multiplied by R will be the Area ; *viz.* RRe. And by the ſame Method of reaſoning, the Area of the other Circle will be rre. But certainly $RRe, rre :: RR, rr :: 4RR. 4rr$. Wherefore, &c.

II. Hence 'tis plain that, the Square of the Diameter of any Circle is to the Area of it : as the Diameter is to $\frac{1}{4}$ Part of the Periphery.

For $$4RR.\ RRe :: 2R.\ \frac{2Re}{4}\left(=\tfrac{1}{2}Re\right)$$

As is plain by multiplying the Extreams and mean Terms by one another.

III. Hence alſo 'tis plain (again) that the Peripheries of Circles are as their Diameters.

That is, $2R.\ 2r :: 2Re.\ 2re.$

IV. And ſince the Area of every Circle is rre, (That is, The Product of the Square of the Radius multiplied into the Name of the Ratio, between its Diameter and Periphery.) A very ready way (for common uſe.) to find the Area of a Circle whoſe Radius is given, will be to multiply the Square of the Radius into this or ſuch like Decimal 3. 1. Thus ſuppoſe the Radius 9 Inches: $81 \times 3.1 = 251.1$. which is very nearly the Area in ſquare Inches, tho' ſomething leſs.

47. All this may be apply'd to Solids. And therefore *Similar Solids* are *ſuch*, as have their Angles all equal, and the ſides about thoſe Angles Proportional ; or (*if they are of a Spherical or of any Spheroidical Figure*) *ſuch*, as can have ſimilar Solids Inſcrib'd or Circumſcrib'd in, or about them, &c.

48. Similar Solids are to one another in a (*Triplicate Ratio* of, or) as the *Cubes* (*of their Homologous Sides*, &c.) See 6. 36, 37, *&c.*

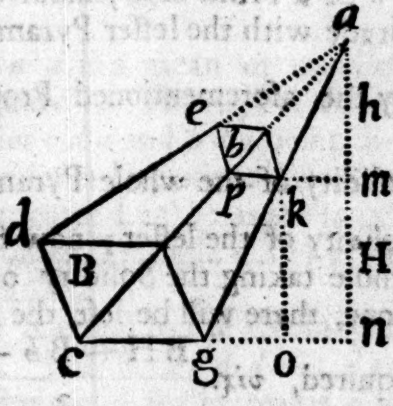
(*And therefore all Spheres muſt be to one another as the Cubes of their Diameters*, &c.) Which may be eaſily thus proved; The Solidity of the Sphere may be expreſſed after this manner; by what is ſaid, in the Corollarys in *p.* 75, 76.

The Area of a great Circle of the Sphere whoſe Radius is R or r, being RRe or rre (by *Cor.* 1. *Art.* 53.) 4 times *that* will be the Surface of each Sphere; That is, $4RRe$ the Surface of the greater, and $4rre$ the Surface of the leſſer: and multiplying the Surface by $\frac{1}{3}$ of the Radius, the Solidities will be $\frac{4RRRe}{3}$ and $\frac{4rrre}{3}$: Which two Quantities being multiplied and divided by the ſame, will be in the ſame *Ratio*, when without ſuch Multiplication and Diviſion: That is $\frac{4RRRe}{3} \cdot \frac{4rrre}{3} :: RRR.\ rrr.$ That is, Spheres are as the Cubes of their Radii, ard conſequently, as the Cubes of their Diameters. Q. E. D.

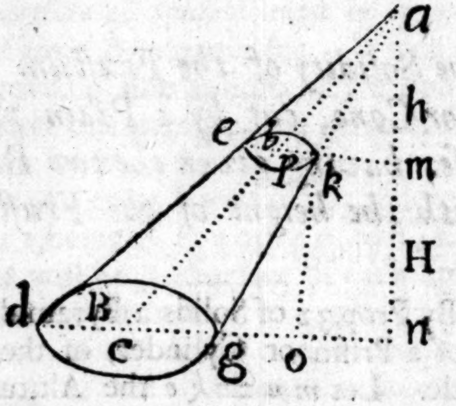
PROB-

PROBLEM.

To find the Solidity of the Fruſtum of a Pyramid or Cone, cut by a Plain parallel to the Baſe, having given the two Baſes together with the height of the Fruſtum.

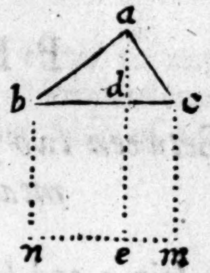


Solution. By Prop. 32 of Solids a Pyramid or Cone is equal to $\frac{1}{3}$ of a Priſm or Cylinder, of the ſame Baſe and Altitude. Let $m\,n = k\,o$ the Altitude of the Fruſtum, be called H, and $m\,a$ the height of the Top piece wanting, b; the Greater Baſe of the Fruſtum B, and the leſſer b; The Triangles $a\,p\,k$ and $a\,c\,g$ are Similar, ($c\,g$ and $p\,k$ being parallel *ex Hyp.*) Wherefore $c\,g .\ g\,a :: p\,k .\ k\,a$, and alternately $c\,g .\ p\,k :: g\,a .\ k\,a$. But $g\,a .\ k\,a :: n\,a .\ m\,a$ (from the Similarity of the Triangles, $g\,a\,n$ and $k\,a\,m$) wherefore *ex equo* $c\,g .\ p\,k :: n\,a .\ m\,a$ and by Diviſion $c\,g - p\,k .\ p\,k :: n\,a - m\,a\ (= m\,n.)\ m\,a$; which put into Symbols (putting $c\,g$ the ſide of the Baſe $=$ S, and $pk = s$) will ſtand thus $S - s .\ s :: H .\ \frac{S\,H}{S - s} = h$. Wherefore having found the Height of the little Pyramid or Cone which is wanting; I ſay, having found it in known Terms, it will be eaſie to find the Solidy of the Fruſtum; for multiplying the Baſe B into the whole height $H + h$, the Product $BH + Bb =$ a Priſm of the ſame Baſe and Altitude with the whole Pyramid or Cone; and



$bh =$ a Priſm or Cylindar of the ſame Baſe and Altitude with the leſſer Pyramid or Cone. Wherefore by the aforementioned Propoſition. $\frac{BH + Bb}{3} =$ Solidity of the whole Pyramid or Cone, and $\frac{bh}{3} =$ Solidity of the leſſer; now from the Solidity of the whole taking the Solidity of the leſſer Pyramid or Cone, there will be left the Solidity of the Fruſtum required, *viz.* $\frac{BH + Bb - bh}{3} =$ Solidity of the Fruſtrum.

The Theorem in words is this; *Multiply the greater Baſe by the whole Height, and from the Product ſubtract the Upper Baſe multiply'd by the Height of the Top piece wanting, and ⅓ of the Remainder will give the Fruſtrum.*



49. If in a Rectangle Triangle *a b c*, a Line as *a d* be drawn from the Vertex or top of the Right Angle, Perpendicular to the Baſe, Hypothenuſe, or longeſt ſide *b c* : it ſhall divide the Triangle *a b c* into two other Rectangled ones, *a b d* and *d a c*, which will be Similar to each other, and to the whole *b a c*. For 1. All the three Triangles have one Right Angle. 2. The Triangles *a b c* and *a b d* have the Angle *b* common to both : Whereſore they are Similar (6. 45). 3. The Triangles *a b c* and *a d c* have alſo the Angle *c* common to both ; therefore they two are Similar ; *and laſtly* a b d *and* a d c *being both Similar to one third Triangle* a b c, *will be ſo to each other*).

50. The Perpendicular *a d* is a mean or middle Proportional between *b d* and *d c*. That is, *c d. d a* : : *d a. d b*. For the Triangles *c d a* and *b d a* being Similar (by the laſt) *c d* (the leſſer Leg of the Triangle *c d a*) ſhall be to *a d* (the greater Leg) : : as the ſame *a d* (the leſſer Leg of the other Triangle *a d b*) is to *b d* the greater Leg. (6. 46).

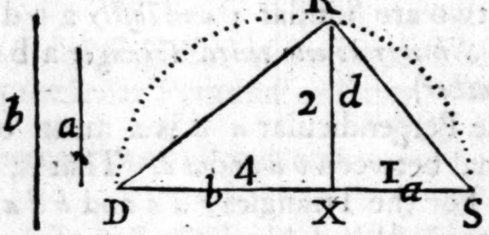
51. The Square of *a d* is equal to the Rectangle made between *c d* and *d b*. For, ſince *c d. d a* : : *d a. d b*. (by the laſt,) the Rectangle of the extreams *c d* and *d b* is equal to the Rectangle of the mean Terms *d a* and *d a* (6. 28). But the two ſides of that Rectangle being equal, becauſe 'tis only *d a* taken twice ; that Rectangle muſt be the Square of *d a* ; and ſo it may be laid down as an univerſal Theorem, that, &c. (*the Square of the Perpendicular drawn from the Vertex of any Rectangle Triangle to the Hypothenuſe, is equal to the Rectangle under the Segments of that Hypothenuſe.*

52. The Square of a mean Proportional is always equal to the Rectangle of the Extreams.

PROBLEM I.

Between two given Lines a *and* b, *to find a mean Proportional, as* d.

Join *a* and *b* both in one Line which make the Diameter of a Circle; and then at the Point *x*, where the given Lines join, erect a Perpendicular as *d*; that



ſhall be the mean Proportional required. For the Angle D R S being a right one (as being in a Semicircle) *b*. *d* :: *d*. *a*, by Prop. 57.

PROB. II.

And thus may you find a Line equal to the Square Root of any Number or Quantity, by finding a mean Proportional between it and 1. For if $b = 4$, and $a = 1$; then will $d = 2$, equal to the Square Root of *b*.

PROB.

PROB. III.

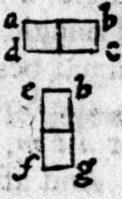
Thus alſo may a Square be found equal to any Rectangle given, by finding a mean Proportional between its Sides, which ſhall be the Side of the Square required.

PROB. IV.

To find a Square equal to any Triangle.

Find a mean Proportional between a Perpendicular let fall from any Angle to an oppoſite Side, and the half of that Side; and that ſhall be the Side of the Square required.

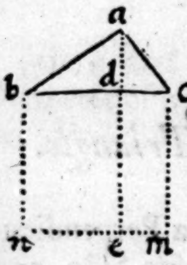
53. *A Rectangle being given to make another Rectangle equal to it, which ſhall have a length given.* Let the Rectangle given be *a c*, and let it be required to make another equal to to it, the length of one of whoſe ſides ſhall be the Line *e f*: here are now three Lines given, *viz.* *a b* and *b c* (which are the ſides of the Rectangle given) and *ef*, which muſt be one ſide of the Rectangle required. Therefore a fourth Line muſt be found, which ſhall be the other ſide of the Rectangle ſought: which is done by finding a fourth Proportional to the three given Lines (6. 43) which let be *e h*. So that *e f*. *a b* :: *b c*. *e h*. and then I ſay, the Rectangle *f h* is equal to *d b*, and is the Rectangle required. (6. 27).



N. B., *This is called* Application *of the Rectangle, equal to a right Line* a b, vid. Eucl. p. 6, e. 6.

54. To expreſs a Rectangle you need uſe but three Letters. *v. gr.* When we ſay the Rectangle *b d c*: We mean a Rectangle one of whoſe ſides is *b d*, and the other *d c*. But if we ſay the Rectangle *b c d*, we then mean a Rectangle one of whoſe ſides is *b c*, and the other *c d*.

55. In every Rectangle Triangle the Square of the Hypothenuſe is equal to the (*Sum of the*) Squares of the two other ſides (*or to the Sum of the Squares of the Legs*).



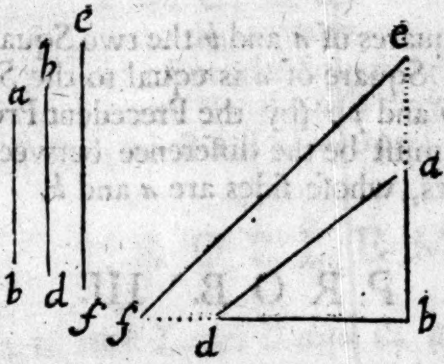
Let the Square *b m*, be divided by the Perpendicular *a d e* into the two Rectangles *d m* and *d n*. I ſay that the Rectangle *d m* is equal to the Square of *a c*, and the Rectangle *d n*, to the Square of *a b*: and that by conſequence the whole Square *b m* is equal to the Sum of the Squares of *a b* and *a c*. For 1. The two Triangles *a d c* and *a b c* being Similar (6. 56). *d c. c a* (in the leſſer Triangle *d c a*) :: as the ſame *a c. c b* in the greater Triangle *a c b*. Wherefore *a c* is a mean Proportional between *d c* and *c b* (or *c m*) and conſequently the Square of *a c* is equal to the Rectangle *b c d*, or *d c m*, that is *d m*.

And after the very ſame manner may *a b* be prov'd to be a mean Proportional between *b d* and *b c* (that is *b n*, &c.) (*for the Triangle* a b d *being Similar to* a b c; d b *the leſſer ſide in one will be to* b a *the greater ſide; as that* b a (*now the leſſer ſide in the other Triangle* a b c) *is to* b c *the greater ſide: That is* d b. b a :: b a. b c. (*or* b n) *and conſequently the Square of* a b *is equal to the Rectangle* d b n, *or* d n. *And ſo both the Squares together, of* b a, *and* a c, *or their Sum, is equal to the Square of the Hypothenuſe.* Q. E. D.

PROB.

PROBLEM I.

Hence any two, or more Squares may eaſily be added together into one Sum. Let *a b*, *b d*, and *e f*, be the Sides of three given Squares, place *a b* and *b d*

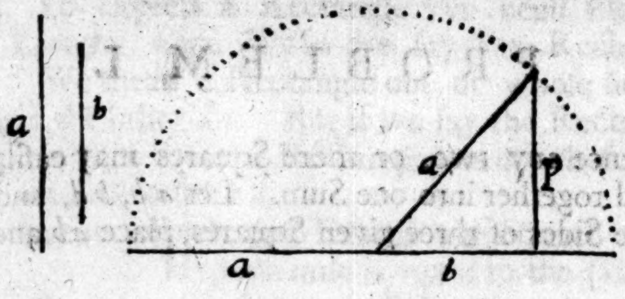


at Right Angles, and draw the Hypothenuſe *a d*, whoſe Square will be equal to the Sum of the Squares of *d b* and *a b*. Then ſet *d a* from *b* to *e*, and the given line *e f*. from *b* to *f*; So ſhall the Hypothenuſe *f e*, be the ſide of a Square equal to the Sum of the three given Squares.

PROB. II.

Or if two Squares be given, you may Subtract one from the other, and find a Square equal to the Difference between them.

Let *a* and *b* be the Sides of the given Squares; make (the longeſt Line) the Radius of a Circle, and ſet *b* from the Center on the ſame Right Line with *a*; at the end of *b*, erect the Perpendicular *p*, which will be the ſide of a Square equal to the Difference be-



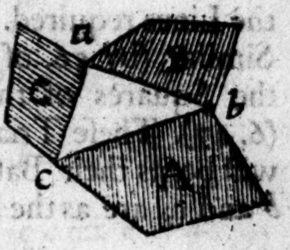
tween the Squares of *a* and *b* the two Squares given; for ſince the Square of *a* is equal to the Sum of the Squares of *b* and *p*: (by the Precedent Prop.) The Square of *p* muſt be the difference between the two given Squares, whoſe ſides are *a* and *b*.

PROB. III.

Hence alſo may a Square be made equal to any given Polygon, or irregular Right-lined Figure: By reducing the Figure into Triangles; finding Squares equal to thoſe Triangles; and then one Square at laſt equal to the Sum of all thoſe Squares.

Or by making Rectangles equal to thoſe Triangles, which ſhall have all the ſame height; then joining thoſe Rectangles together, ſo as to make one great one equal to them all; and laſtly make a Square equal to that Rectangle.

59. If upon the three ſides of a Rectangle Triangle are made three ſimilar Figures, and thoſe ſimilarly Poſited, the greateſt ſhall be equal to the other two.

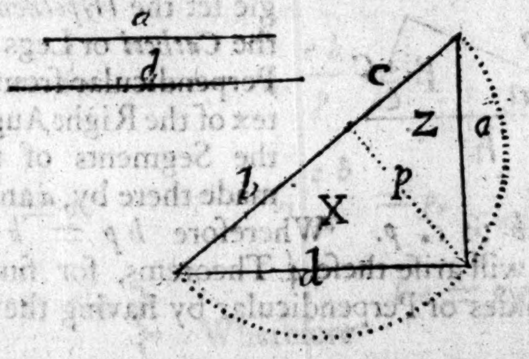


For the three Figures being Similar are as the Squares of their Homologous ſides (6. 53). And therefore the Figure A ſhall be to B and C, as the Square of *b c* is to the Squares of *a b* and *a c*. But the Square of *b c* is equal to thoſe two Squares (*by the laſt*) therefore (*the Figure* A *is equal to both* B *and* C *together*).

PROBLEM I.

To find two Lines b *and* c, *which ſhall have the ſame Ratio to one another, as two given Squares, Similar Triangles, Similar Polygons, or Circles.*

Let *a* and *d* be the Sides of the two given Squares, Triangles, Polygons; or the Diameters or Radius's of the Circles given: Set them at Right Angles to

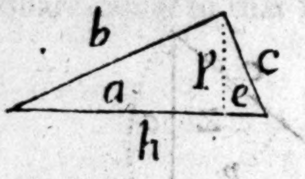


one another, as you ſee, and draw the Hypothenuſe $b+c$, to which let fall the Perpendicular p, which ſhall divide the Hypothenuſe into two Parts b and e, the Lines required. For the Triangles Z and X being Similar (by 56. of the 6), will be to one another as the Squares of their Homologous Side a and d, (6. 47). Theſe Triangles alſo having the ſame height, will be as their Baſes (6. 42), wherefore their Baſes b and c, are as the Squares of d and a. Q. E. D.

PROB. II.

This Problem may be Inverted thus; To make two Squares, Triangles, &c. *having the Ratio of two given Lines,* b *and* c, *or in any given Ratio.*

Join the Lines into one continued Line, and then make that the Diameter of a Circle, from the point where b and c join, erect a Perpendicular to the Curve as p, then draw d and a, and they ſhall be the Sides of the Squares, Triangles, ſimilar Polygons, or the Diameters of the Circles required.



In a Right angled Triangle let the *Hypothenuſe* be b the *Catheti* or Legs b and c, a Perpendicular from the Vertex of the Right Augle p, and the Segments of the Baſe made there by, a and e. Then

1. $h.\ b :: c.\ p.$ Wherefore $hp = bc$, from whence will ariſe theſe 4 Theorems, for finding any of the Sides or Perpendicular by having the reſt.

$$\left.\begin{array}{l} 1.\ h=\frac{bc}{p}. \\[2ex] 2.\ b=\frac{hp}{c}. \end{array}\right\} \qquad \begin{array}{l} 3.\ c=\frac{hp}{b} \\[2ex] 4\ p=\frac{bc}{h} \end{array}$$

$$\left.\begin{array}{l} 2.\ a.b :: b.h \\ e.c :: c.h \end{array}\right\} \text{ Wherefore } \left.\begin{array}{l} bb=ah \\ cc=eh \end{array}\right\}$$

Whence will ariſe theſe two Theorems for finding the Segments from the 3 ſides.

$$1.\ \frac{bb}{h}=a. \qquad 2.\ \frac{cc}{h}=e.$$

$$\left.\begin{array}{l} 3.\ a.p :: b.c \\ e\,p :: c.b \end{array}\right\} \text{ Wherefore } \left\{\begin{array}{l} ac=pb \\ eb=pc \end{array}\right\}$$

From whence theſe Theorems will ariſe for finding the Segments, the Sides or the Perpendiculars.

$$1.\ \frac{pb}{c}=a. \qquad 2.\ \frac{pc}{b}=e.$$

$$3.\ \frac{ac}{p}=b. \qquad 4.\ \frac{eb}{p}=c.$$

$$5.\ \frac{ac}{h}=p. \qquad 6.\ \frac{eb}{c}=p.$$

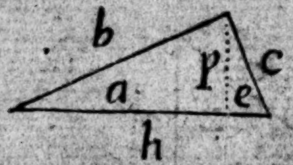
$$\left.\begin{array}{l} 4.\ b.p :: c.e \\ c.p :: b.a \end{array}\right\} \text{ Wherefore } \left.\begin{array}{l} be=pc \\ ca=pb \end{array}\right\}$$

one another, as you ſee, and draw the Hypothenuſe $b+c$, to which let fall the Perpendicular p, which ſhall divide the Hypothenuſe into two Parts b and e, the Lines required. For the Triangles Z and X being Similar (by 56. of the 6), will be to one another as the Squares of their Homologous Side a and d, (6. 47). Theſe Triangles alſo having the ſame height, will be as their Baſes (6. 42), wherefore their Baſes b and c, are as the Squares of d and a. Q. E. D.

PROB. II.

This Problem may be Inverted thus; To make two Squares, Triangles, &c. *having the Ratio of two given Lines,* b *and* c, *or inany given Ratio.*

Join the Lines into one continued Line, and then make that the Diameter of a Circle, from the point where b and c join, erect a Perpendicular to the Curve as p, then draw d and a, and they ſhall be the Sides of the Squares, Triangles, ſimilar Polygons, or the Diameters of the Circles required.



In a Right angled Triangle let the *Hypothenuſe* be b the *Catheti* or Legs b and c, a Perpendicular from the Vertex of the Right Augle p, and the Segments of the Baſe made there by, a and e. Then

$1.\ h.\ b :: c.\ p.$ Wherefore $bp = bc$, from whence will ariſe theſe 4 Theorems, for finding any of the Sides or Perpendicular by having the reſt.

$$\left.\begin{array}{l} 1.\ h = \dfrac{bc}{p}. \\[2ex] 2.\ b = \dfrac{hp}{c}. \end{array}\right\} \quad \begin{array}{l} 3.\ c = \dfrac{hp}{b} \\[2ex] 4\ p = \dfrac{bc}{h} \end{array}$$

$$\left.\begin{array}{r} 2.\ a.b :: b.h \\ e.c :: c.h \end{array}\right\} \text{ Wherefore } \left.\begin{array}{l} bb = ah \\ cc = eh \end{array}\right\}$$

Whence will ariſe theſe two Theorems for finding the Segments from the 3 ſides.

$$1.\ \frac{bb}{h} = a. \qquad 2.\ \frac{cc}{h} = e.$$

$$\left.\begin{array}{r} 3.\ a.p :: b.c \\ e\ p :: c.\ b \end{array}\right\} \text{ Wherefore } \left\{\begin{array}{l} ac = pb \\ eb = pc \end{array}\right\}$$

From whence theſe Theorems will ariſe for finding the Segments, the Sides or the Perpendiculars.

$$1.\ \frac{pb}{c} = a. \qquad 2.\ \frac{pc}{b} = e.$$

$$3.\ \frac{ac}{p} = b, \qquad 4.\ \frac{eb}{p} = c.$$

$$5.\ \frac{ac}{b} = p. \qquad 6.\ \frac{eb}{c} = p.$$

$$\left.\begin{array}{r} 4.\ b.p :: c.e \\ c.\ p :: b.\ a \end{array}\right\} \text{ Wherefore } \left.\begin{array}{l} be = pc \\ ca = pb \end{array}\right\}$$

And conſequently,

1. $e = \frac{pc}{b}$ 2. $a = \frac{pb}{c}$.

5. And ſince $e = \frac{pc}{b}$ and (by 3) $\frac{cc}{b} = e$.

Therefore $\frac{pc}{b} = \frac{cc}{b}$. Wherefore $pcb = bce$ and dividing both by c, $pb = be$.

That is the Rectangle under the Leggs, is equal to that of the Perpendicular into the Hypothenuſe, &c.

For by proceeding after this Method the Reader may eaſily diſcover many ſuch Propoſitions as theſe: Which I leave to exerciſe his Skill and Diligence this way.

I. That the Rectangle under either Leg of a right Angled △, and the oppoſite Segment of the Baſe, is equal to that under the Perpendicular into the other Leg.

II. The Square of the Hypothenuſe is to that of either Leg :: as the Rectangle under the Hypothenuſe, and the Segment of its oppoſite to that Leg is to the Square of the Perpendicular.

III. The Solid under the Perpendicular into the Rectangle of the Legs is equal to that under the Hypothenuſe into the Rectangle of its Segments.

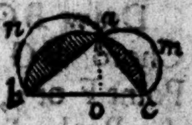
IV. The Square of the Perpendicular, is to the Square of an Leg, as the Segment oppoſite to the Leg, is to the whole Hypothenuſe.

VI. The

V. The Square of one Leg into the oppoſite Segment of the Hypothenuſe equal to the Square of the others into its oppoſite Segment, Wherefore,

VI. The Squares of the Sides are as thoſe Segments.

56. If on the Hypothenuſe *b c* of a Rectangle Triangle, there be made a Semicircle *b a c*, and on the other two Sids *a b* and *a c*, two more Semicircles *b n a* and *a m c*, that great Semicircle will be equal to the other two (by the laſt Propoſition.) And if from the greater Semicircle, and the two leſſer ones you take away what is common to both; which are the two ſhaded Segments *a b* and *a c*; what remains of each muſt be equal, *i. e.* the Triangle *a b c* is equal to both the Lunes *b n a* and *a m c*.



And this is the Quadrature of the *Lunes* of *Hippocrates* of *Scio*.

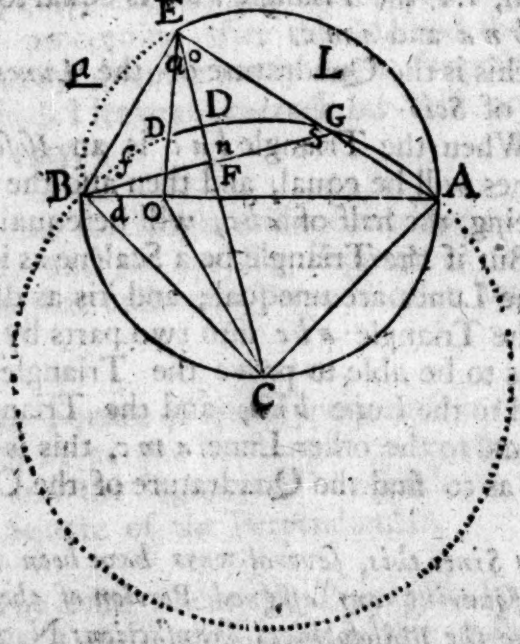
57. When the Triangle *b a c* is an *Iſoſcles*, then the Lunes will be equal, and then alſo the Triangle *a b o*, being the half of *a b c*, will be equal to each Lune, But if the Triangle be a Scalene as in this Figure, the Lunes are unequal; and 'tis as difficult to divide the Triangle *a b c* into two parts by the Line *a o*, ſo as to be able to prove the Triangle *a b o* to be equal to the Lune *b n a*, and the Triangle *o a c* to be equal to the other Lune *a m c*, this is I ſay, as difficult as to find the Quadrature of the Circle.

N. B. *Since this, ſeveral ways have been diſcovered of ſquaring any aſſigned Portion of theſe Lunes* (*See the Philoſophical Tranſactions* N. 259. pag. 4, 11). *Of which this is one.*

Let there be a greater Circle B G A C on whoſe Quadrantal Arch B A let the Lune B E A G B, or L, be drawn by deſcribing the Semicircular Arch B E A, which is one half of the leſſer Circle B C A E. Let then a Line as C E be drawn from the Center of the greater Circle cutting off any Portion or Segment of the Lune, as B E D: 'Tis required to Square that Segment.

Draw B G at right Angles to E C; So ſhall the Chord B G be Perpendicularly Biſſected in the Point F or *n*, draw alſo B E and E G A. I ſay, that the Right lined Triangle B E F is equal to the Part of the Lune B E D.

For F G being equal to F B, E F common to both, and the Angles at F equal, becauſe both Right, the



Triangle E F G will be equal to B E F: Wherefore the Angle *o* being equal to *a*, they muſt be both Semi-right; And conſequently, *f* and S muſt be alſo

Semi-right: Therefore the three Triangle E B G, E B F and E F G muſt be each one the half of a Square. And conſequently, G B. E B :: √: 2. √: 1. for the Square of G B is double the Square of E B, and ſince Similar Segments are as the Squares of their Chords, the Segment B G muſt be double of B E: Wherefore the half of one will be equal to all the other; that is, B D F equal to the Segment B E. And therefore the Rectilineal Triangle B E F, exceeding the Portion of the Lune by the half Segment BDF, and falling ſhort of the Lune by the Segment BE, which is equal to that former half Segment B D F, the Triangle is exactly equal to the Portion of the Lune. Q. E. D.

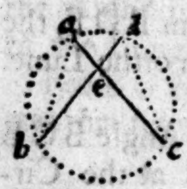
And the ground of all is this, that the Angle B C E being at the Center of one Circle, and at the Circumference of the other, muſt divide the Quadrantal Arch B G A in the ſame Proportion as it doth the Semi-circular one B E A: On which depends the Equality of the Segments B E and B D F.

And ſince the Triangle B C A is equal to the Lune L (as is apparent by taking the common Segment B G A B from the Semi-circle B E A B and from the Quadrant B G A C.) It will be eaſie to take from thence a part, as the Triangle B O C equal to the Aſſigned Portion of the Lune. For having let fall a Perpendicular from E to find the Point O, draw O C; and then will the Triangle B O C be equal to the Triangle B E F, before proved equal to the Segment of the Lune. For the Triangles B C A and B E F are ſimilar, as being each the half of a Square: And therefore the former to the latter will be as the Square of B A. to the Square of B E (6. 47) their Homologus Sides. That is, as B A. is to B O (6. 30) for B E is a mean Proportional between B A and B O. Farther, the Triangle B A C, having the ſame Height with B O C, will be to it as the Baſe A B to B O.

Wherefore the two Triangles BEF and BCO, being proved to have the ſame Ratio, to one and the ſame thing muſt be equal. Q. E. D.

And therefore to divide the Lune according to any given Ratio, you need only divide the Diameter A B according to that Ratio in the point O, and from thence erect a Perpendicular to find the point E; then draw E C. which ſhall cut off the aſſigned Portion of the Lune.

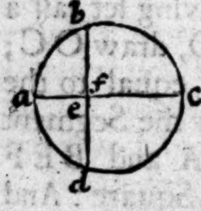
58. Two Chords cutting or croſſing each other in a Circle have the Segments *Reciprocally Proportional.*



I ſay that *a c. b e* : : *d c. e c.* and conſequently the Rectangle *a e c* is equal to the Rectangle *d e b.*

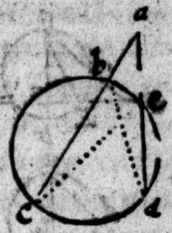
For draw the prickt Lines *a b* and *d c* and the two Triangles *a b e* and *d c e* will be Similar; Becauſe 1. The vertical or oppoſite Angles at *e* are equal (1. 23). 2. The Angle *b* is equal to *c,* becauſe ſtanding both on the ſame Ark *a d,* and being in the ſame Segment (4. 12). wherefore the two Triangles are ſimilar, and conſequently *a e. e b* : : *d e. e c.* (6. 46). Q. E. D.

59. If *a c* be the Diameter of a Circle, and *b d* a Perpendicular to it, *d e* or *b e* will be a mean Proportional between the Segments of the Diameter *a e* and *e c,* Becauſe *d e* is equal to *e b* (by 4. 6). and therefore ſince (*by the laſt*) the Rectangle *b e d* (that is *b e* Square) is equal to *a e c,* as the Rectangles of the Parts of all croſſing Chords are; the Line *b e* or *e d* muſt be a mean Proportional between *a e* and *e c.* Q. E. D.

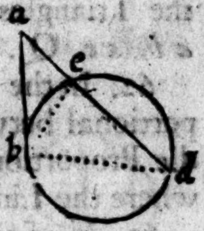


60. Two

66. Two Lines ac and ad drawn from a Point a without a Circle, to the Internal and oppoſite part of its Circumference; are to each other *Reciprocally* as their external Segments. I ſay, ac. ad :: ae. ab, and conſequently the Rectangle cab is equal to dae. For ſuppoſing the Lines ce and bd to be drawn, the Triangles ace and adb will be ſimilar, becauſe the Angle a is common to both, and the Angle c is equal to d becauſe ſtanding on the ſame Ark be (4, 12), wherefore da. ab :: ca. ae, and Alternately da. ca :: ab. ae, and by Inverſion ca. da :: ae. ab (6. 45). And therefore the Rectangle cab is equal to dae. Q. E. D.

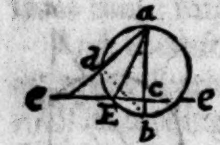


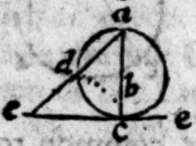
If one Line as ab touch a Circle, as in the Point b, and another Line ad drawn from the ſame point a, do cut it; Then is ab (*the Tangent*) a mean Proportional between ad and ae (i. e, *between the whole Secant and the Part of it without the Circle.*)

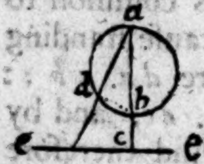


For drawing the Lines be and bd the Triangles aeb and bad will be Similar; becauſe the Angle a is common to both, and the Angle abe (made by the Tangent, and Secant eb) is equal to d (*an Angle in the oppoſite Segment*) (4. 17). therefore they are ſimilar and conſequently ea (in the little Triangle) will be to ab :: as that ſame ab is to ad in the greater Triangle; i. e, ea. ab :: ab. ad. Q. E. D.

61. Let there be a Diameter *a b* cut in *c* by an Infinite Perpendicular *e e*, whether within the Circle as in *Fig.* 1, at the Circumference as in *Fig.* 2. or without the Circle as in *Fig.* 3. Let there be drawn alſo from the Point *a* any Right Line as *a e* cutting the Perpendicular in *e* and the Circle in *d*. I ſay it ſhall always be as *a d. a* : : *a b. a e.*



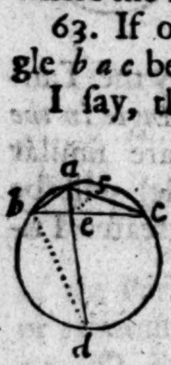




For drawing the Line *b d*, there will be made two Triangles that are Similar, as *e a c* and *d a b*; which will be ſo, becauſe they have one Angle as *a* common to both, and the Angle *d* equal to *c*, becauſe both are Right ones (for *d* is Right by 4. 14), as being an Angle in a Semicircle and *c* is Right by the Suppoſition. Wherefore the Triangles are ſimilar and conſequently *a d. a c* : : *a b. a e.* Q. E. D.

62. In the ſecond Figure *a b* is always a mean Proportional between *a e* and *a d*; in the firſt, the middle Proportional is *a* E, drawn from *a* to the place where the Line *e c* cuts the Circle.

63. If of a Triangle inſcribed in a Circle, the Angle *b a c* be Biſſected by the Line *a e d*.

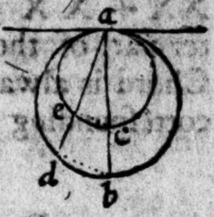


I ſay, then *b a. a e* : : *a d. a c*. For drawing the Line *b d* there will be made two Triangles *a b d* and *a e c*, which are Similar; becauſe the Angle *d* is equal to *c* (4. 12), as (*being in the ſame Segment*) or inſiſting on the ſame Ark, and *b a d* is equal to *e a c* by the *Suppoſition*. Wherefore the Triangles are Similar, and conſequently *b a. a d* : : *a e. a c.* (and *therefore alternately* b a. a e. : : a d. a c.) Q. E. D.

64. When the Angle at the Vertex is thus biſſected, the Segments o fthe Baſe *b c*, re alſo proportiona lto the Legs of the Triangle (*i. e.*) *be. ec* .: *b a. a c.* For ſuppoſing *e f* drawn Parallel to *b a.* Then will *b a. ac* :: *ef. f c* (6. 40). But *ef* is equal to *a f*: becauſe the Angle *a e f* is equal to *e a b*, (*as being Alternate Angles* 1. 31), and conſequently to *e a f* (*by the Suppoſition*) wherefore the Triangle *a e f* is an *Iſſocles* (2. 15.) And therefore inſtead of putting of it as before *b a. a c* :: *e f. f c*, we may ſay *b a. ac* :: *a f. f c.* But as *a f. f c* : : ſo is *b e. e c* (6. 42). wherefore *b a. a c* :: *b e. e c.* Or, which is all one *b c. e c* :: *b a. ac.* Q. E. D.

(N. B. *This Propoſition is Univerſal ; and if any Angle of a Triangle be Biſſected, the Legs about that Angle are proportional to the Segments of the oppoſite ſide made by the Line Biſſecting the Angle.*)

65. If two Circles touch one another (*in a Point within*) as *a*, and if to that Point you draw a Tangent and a Perpendicular *a c b* (wnich will paſs through both their Centers) (4. 5.) and if alſo you draw any Secant from the ſame Point as *a e d.* I ſay, 'twill always be as *ae. a d* : : *a c. a b.* For having drawn the Lines *e c* and *d b*, the Triangles *a e c* and *a d b* will be Similar, as having the Angle at *a* common; and *e* and *d* both right ones; (by 4. 14). and conſequently *a e. a d* :: *a c. a b.* Q. E. D.

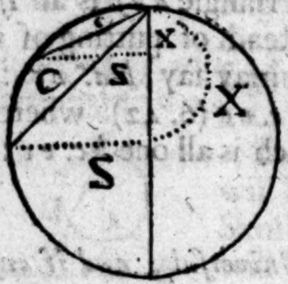


66. The Ark *e c* is to the Ark *d b* as the whole Circle *a e c* to the whole Circle *a d b.* (6. 49 and 4. 11.)

PROP.

PROP.

67. If two (or more) Chords, as c, C, iſſue from the ſame end of any Diameter of a Circle. *Their Squares ſhall be directly, as the Verſed Sines* x. X.



And ſhall alſo be equal to the Rectangles under the Diameter and ſuch Verſed Sines.

Let fall the right Sines s and S. Then will $cc = ss + xx$ and $CC = SS + XX$, and if the Diameter be called D, its parts will be x and $D - x$, X and $D - X$ But $ss = xD - xx$, and $SS = DX - XX$ (by 66 of this Book) wherefore Subitute thoſe two laſt Quantities inſtead of the equals ss and SS; and you will have $cc = Dx - xx + xx$ (that is) Dx and $CC = DX - XX + XX$ (that is) DX which proves the latter part of the Propoſitition that the Square of the Chord is always equal to the Rectangle under the correſponding verſed Sine, and the whole Diameter.

And 'tis plain that,

$$Dx . DX :: x . X.$$ Q. E. D.

PROP.

A Circle whoſe Area is equal to the Convex Surface of a given Cone, will have its Radius a mean Proportional between the ſide of the Cone and Radius of its Baſe.

Let the Side of the Cone be $= a$, the Radius of the Baſe $= r$; then the Diameter will be $2r$, and the Peripherie $= 2re = c$. But half the Periphery into the Side of the Cone is $=$ to the Convex Surface of the Cone (by) that is are expreſſes the *Area* of the Cone. Now ſince $\sqrt{:ar}$ is a mean proportional between a and r (for $a . \sqrt{:ar} :: \sqrt{:ar} . r$) I Imagine $\sqrt{:ar}$ to be the Radius of the Circle whoſe Area $=$ Area of the Cone. Then will its Diameters be $2\sqrt{:ar}$ and its Periphery $2\sqrt{:are}$: and by Multiplication of $2\sqrt{:are}$, the Periphery into $\frac{1}{2}\sqrt{:ra}$ the half Radius: or $\sqrt{:are}$ into $\sqrt{:ra}$ the Radius of the Circle will be $are = b$ the Surface of the Cone. Q. E. D.

69. The Convex Surface of a right Cone is to the Area of its Baſe :: as the Side of the Cone is to the Radius of the Baſe.

For ſince the Convex Surface of the Cone, (by what is ſaid after Art. 14. Book. 4.) is equal to a Triangle whoſe Baſe is equal to the Periphery of the Circular Baſe of the Cone, and its height the Side of that Cone, call the Periphery c and the Side of the Cone a, then will $\frac{ac}{2}$ expreſs the Area of the Convex Surface, and the Area of the Baſe will be $\frac{rc}{2}$. (by Art 26. Book 4.) But there is no doubt but $\frac{ac}{2} . \frac{rc}{2} :: a . r$. Wherefore, *&c.*

70. A Circle whoſe Radius is equal to the Diameter of the Sphere will have its Area equal to the Spheres Surface.

Let the Radius of ſuch a Circle be $2r$, then its Diameter is $4r$, and its Periphery will be $4re$ and by multiplying that by $r =$ half of Radius, the Area is $4rre$. Let then the Radius, of the Sphere be r, then will its Diameter be $2r$, and the Periphery of a great Circle $2re$, which being multiplied by the Radius r, makes $2rre$; the half of which is rre, the Area of a great Circle, but the Area of 4 ſuch Circles is equal to the Spheres Surface (by Cor. V. *p* 76) that is $4rre =$ to the Spherse Surface; which was above proved equal to the Area of the Circle whoſe Radius was equal to the Spheres Diameter. Wherefore, *&c.*

ELEMENTS OF GEOMETRY.

BOOK VI.

Of Incommenſurables.

A Leſſer Quantity is ſaid to *Meaſure* a greater, when being taken a certain number of Times, it is exactly equal to the greater. *v. gr.* Suppoſe a Fathom to contain ſix Feet; then may one Foot be ſaid to *Meaſure* that Fathom, becauſe being taken or repeated ſix times, it will be exactly equal to the Fathom.

2. The Quantity which is thus a *Meaſure* to a greater Quantity, is called a *Part* of that greater; and the greater Quantity is call'd the *Multiple* of the leſſer. So, a Foot is the *Part* of a Fathom, and a Fathom is the *Multiple* of a Foot.

3. If you take the Quantity (*of a common* French *Pace*) which is two Foot and a half, and try with that

to meaſure a Fathom, you cannot do it: Becauſe if you add that Pace only twice, it will make but five Foot, which are leſs than the Fathom; and if you take it three times, it makes ſeven Foot and a half, which are more than the Fathom; ſo that this Quantity of two Foot and an half cannot meaſure the Fathom, and therefore properly ſpeaking is not a *Part* of it. But nevertheleſs they may be ſaid to be *Parts* of the Fathom, becauſe this Quantity contains five half Feet; for an half Foot is a *Part* of a Fathom, becauſe being taken 12 times it will juſt Meaſure it; ſo therefore this Pace contains *Parts* of the Fathom, becauſe it contains five half Feet which are $\frac{5}{12}$, that is five twelfths of a Fathom.

4. When two Quantities are ſuch, that a third can be found which ſhall be an (*Aliquot or Even*) Part of both, that is, which ſhall *meaſure* them both exactly: Then thoſe Quantities are ſaid to be commenſurable: As for inſtance, a Pace and a Fathom are two commenſurable Quantities, becauſe we can find a third Quantity, *viz.* half a Foot, which will meaſure them both; For if the half Foot be taken five times it makes the Pace, and taken 12 times, it makes the Fathom.

5. But when it is not poſſible to find any third Quantity which can meaſure two others, then thoſe two Quantities are called Incommenſurables.

6. Commenſurable Quantities are as *Number to Number*, that is, thoſe Quantities can be expreſſed by Numbers, ſo that as one Quantity is to the other, ſo ſhall one Number be to the other. Thus a Line of ſix Feet or a Fathom, and a Line of two Foot and an half, as a Pace, are to one another as Number to Number. For half a Foot meaſuring them both, the latter by being taken 5 times, and the former by being taken 12 times; it's plain that one Line contains 5 half Feet, and the other 12, and therefore they are as 5 to 12, or as Number to Number.

7. If two Quantities are not as Number to Number, that is, if it be impoſſible to expreſs their Magnitudes by two Numbers, they are Incommenſurable: As is plain from the laſt.

8. We ought then to ſee whether there are in Reality any ſuch Quantities whoſe Magnitudes cannot be expreſs'd by Numbersand if there be any ſuch, we muſt ſay that there are *Incommenſurable Quantities.*

9. A *plane Number* is that which may be produced by the Multiplication of two Numbers (*one into another*). *v. g.* 6 is a plane Number, becauſe it may be produced by the Multiplication of 3 by 2: For twice 3 makes 6: So alſo 15 is a plane Number ariſing from 5 being multiplied by 3; and 9 is a plane Number produced by the Multiplication of 3 by 3.

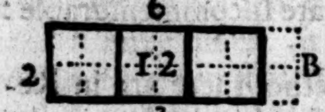
10. Thoſe Numbers which being multiplied one by another, do produce a plane Number, are called the *Sides* of that Plane, as 2 and 3 are the ſides of the Plane 6; and 3 and 5 are the ſides of 15.

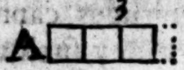
11. If we Imagine Unites to be little Squares, thoſe Squares may be form'd into a Rectangle, if their Number be a Plane. *v. gr.* 12 Squares may be placed in the form of a Rectangle, one of whoſe ſides may be 6 and the other 2, and 48 will make a Rectangle whoſe two ſides may be 12 and 4. See the following Figures B and C.

12. *A Square Number* is a plane, whoſe ſides are equal; as 4, ariſing from the Multiplication of 2 by 2; as 9, the Product of 3 by 3: and 16 made by 4 multiplied by 4, *&c.*

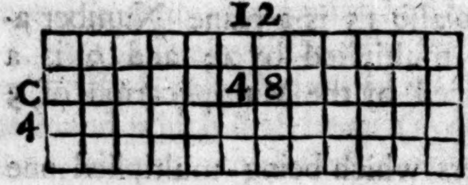
13. A Square Number may be ranged into the form of a Square and that Number which can be ranged into the Form of a Square is a Square Number, and that which cannot be ranged into the Form of a Square, is not a Square Number.

14. *Similar plane Numbers* are thoſe which may be ranged into the Form of ſimilar Rectangles; that is into Rectangles, whoſe Sides are proportional; ſuch are 12 and 48; For the Sides of 12 are 6 and 2 (*See Fig.* B) and the Sides of 48 are 12 and 4 (*See Fig.* C.) But 6. 2 :: 12. 4. and therefore thoſe Numbers are Similar.





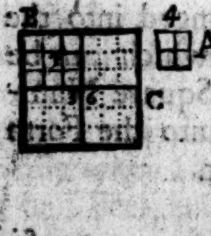
15. All ſquare Numbers are ſimilar Planes (6. 32).



16. Every Number may be placed in the Form of a Right-line, and in that diſpoſition may be taken for a Plane. Thus 3 (*in Fig.* A) may be conceived as a Plane Similar to 12 or B; For the ſides of the Plane 3, are 1 and 3, (becauſe once 3 is 3) and the ſides of 12 are 2 and 6. But as 1. 3 :: 2. 6.

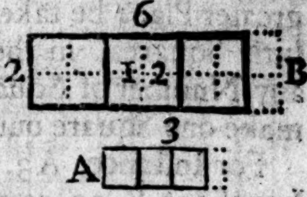
17. There are Numbers which are not *Similar Planes*. As if you examine from 1. to 10, you will find indeed that 1. 4. 9 being Squares are ſimilar, and ſo are 2 and 8, which have one ſide double to the other. But the reſt as 3. 5. 6. 7. are by no means ſimilar Planes.

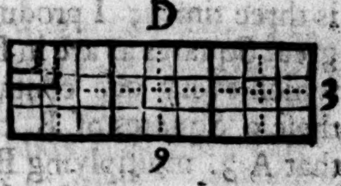
18. If one Square Number be multiplyed by another, the Product will be a third Square Number. Thus A. 4. and B. 9. being both Squares do, when multiplied into one another, produce the Number 36 or C: And I ſay that third Number is a Square. For the meaning of multiplying B by A, is to take B as often as there are Unites in A. But



But I may conſider the whole Number B. 9. as one only Square, and I can take that as often as there are Unites or little Squares in A. And as the Unites in A, are ranged into a Square, ſo I can range the Square B as often into a Square Form, juſt as if it were an Unite. So that there will be four ſuch Squares of B, which being placed as you ſee in the Figure will make the Square C or 36.

19. If two Numbers are ſimilar Planes, the greater may be divided into as many Squares as their are Unites in the leſſer. A. 3. and B. 12. are ſimilar Planes; ſo that the ſide 3. is to 6 :: as the ſide 1. is to 2 Wherefore I can divide the Plane B. 12. into 3 Squares placed juſt in ſuch a manner as thoſe 3 little Squares in the Plane A. And every one of the great Squares of B ſhall anſwer to 4 of thoſe in A. So alſo if rhe Planes had been 8 and 72; I can Divide 72 into eight Squares of which every one ſhall contain 9 of thoſe in the leſſer Plane 8. The ſame would come to paſs alſo, if either one, or both the Numbers had been Fractions. As if A contain 3 and $\frac{1}{2}$, and B 14. I can divide 14 into three Squares and an half, diſpoſed juſt like thoſe in A. as may be ſeen by the Partitions in the Figure, and by the half Square added in Prickt Lines.



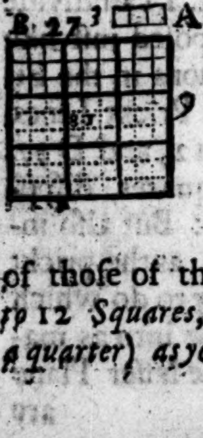


In like manner if the Planes were B 12, and D 27, I can divide 27, not only into three Squares, diſpoſed after the ſame manner as thoſe in A: But alſo into 12 Squares, ſo ranged as thoſe in B. as the prickt Lines in the Figure D do ſhew. The way to do which is to divide the Sides of the greater Plane into as many Parts as the Homologous ſides of the leſſer Plane are divided into; the Figure ſhews the thing, and makes it eaſie.

20. Thoſe plain Numbers which can be ſo divided as that there are as many Squares in the greater Plane, as there are Unites in the leſſer, are Similar; this is the Converſe of the former.

21. Two ſimilar Plane Numbers multiplied one into another do produce a Square. For having divided the greater Plane into as many Squares as there are Unites in the leſſer (7. 19.) One Plane will be multiplied by the other, if the greater Squares of the greater Plane be taken as often as there are Unites or little Squares, in the leſſer Plane: But to multiply any Number of Squares, by the ſame Number, is to make one Square out of all thoſe Squares.

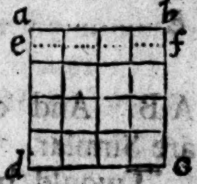
For Inſtance, A 3. and B 27. being Similar Planes, I conſider B. 27 as a Plane compos'd of three great Squares, as A 3. is a Plane compos'd of three Unites, or three little Squares. So that if I take all theſe three great Squares, as often as there are Unites in A, that is three times; I produce then three times three ſuch great Squares as are in B, that is, 9 ſuch Squares; of which every one contains 9 of thoſe in A, and all theſe 9 Squares of B contain 81 of thoſe of A; ſo that A 3. multiplying B 27. produces 81. which is a Number of the leſſer Squares rang'd into a Square Figure; and by conſequence a ſquare Number (7. 13) In like manner if the Planes were B. 12. and D. 27. I divide 27 into 12 Squares, which I multiply by 12, and there are produced 144 greater Squares rang'd into the form of a Square, which do contain in all 324 of thoſe of the leſſer Plane. (N. B. *To divide 27 into 12 Squares, each Square muſt be 2. 25. (or two and a quarter) as you may ſee it is in the Figure* D. N°. 19).



29. If two Plane Numbers are Similar, after what form ſoever you range one, the other alſo may be ſo diſpos'd. Let 3 and 12 be ſimilar Planes. If 12 be ſo rang'd in a Right-line that it will make a Rectangle, one of whoſe ſides ſhall be 12, and the other 1. I ſay that 3 may be ſo diſpoſed as to make a ſimilar Rectangle, one of whoſe ſides will be 6, and the other the half of one, *&c.*

23. If one number divide another that is a Square one, a third ſhall be produced which will be a Plane Similar to the Diviſor.

Let there be a Square *a c* 16, and let it be divided by any Number, as ſuppoſe by 8. which is done if you take the eighth Part of the ſide *a d*, viz. *a e*, and thro' *e* draw the Parallel *e f*. For by that means you will have the Plane *a f*, which will be the eighth Part of the Square *a c*. But to divide a Number or a Plane by 8, is to take the eighth Part of that Number or Plane.



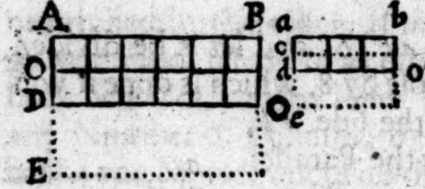
I ſay the Plane *a f*, is ſimilar to 8; for 8 being ranged into a Right-line, ſo as to make a Rectangle, one of whoſe ſides ſhall be 8. and the other 1. ſhall be Similar to it, becauſe *a e* was taken the eighth Part of *a d* or *a b*: Wherefore as 8. 1 : : (which are the ſides of the Plane 8 the Diviſor) ſo ſhall *a b*. *a e* (which are the ſides of the Plane of the Quotient ariſing when the Square *a c* was divided by 8). Therefore if one Number divide another that is a ſquare, *&c* Q. E. D.

24. If two Planes multiplying one another do produce a Square, thoſe Planes are Similar.

25. Two Plane Numbers which are not Similar, if they are multiplyed into one another, cannot produce a Square. Theſe two Propoſitions are Conſectaries from the foregoing ones.

26. If two Numbers are ſimilar Planes, their *Equi-multiples* and any of their (*Reſpectively*) equal Parts, are alſo Similar Planes. Let the Planes be *a b c d.* 3. and AB CD. 12. ſo that *a b*. A B :: *b e*. B C. I ſay if you take the double of the one and the double of the other (or any other Equi-multiple, be it what you pleaſe) thoſe doubles ſhall be Similar.

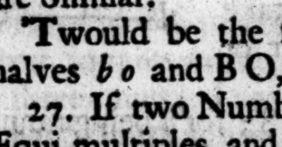
For having taken *a e* double to *a d*, and A E double to A D: in order to make the Plane *b e* double to *b d*, and B E double to B D: 'Tis clear that *a d*. A D :: *a e*. A E. But *a d*. A D :: *a b*. A B. wherefore alſo *a e*. A E :: *a b*, A B. And conſequently the Planes *b e* and B E are Similar.



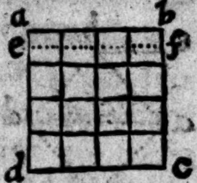
'Twould be the ſame thing had you taken their halves *b o* and B O, or any other equal Parts of each.

27. If two Numbers are not ſimilar Planes, their Equi-multiples, and all their (*reſpectively*) equal Parts will alſo be not Similar, which follows from the laſt.

28. Between any two ſimilar plane Numbers whatſoever, there is to be found a mean Proportional. Let the two Nnmbers be 2 and 8, I ſay it is poſſible to find a Number which ſhall be a mean Proportional between them. For if we imagine the Plane 8 to be ranged in a right line A B, and the Plane 2, alſo to be ranged in another Right-line as a A D. and that out of thoſe two Right-Lines there be formed the Plane A C. 16. That Plane A C. 16. will be produced by the Multiplication of the two Numbers 2 and 8 (6. 17 and the following Propoſitions) and conſequently the Number of the little

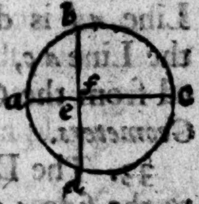


tle Squares of the whole Plane A C. 16. ſhall be a ſquare Number (7. 21) and they may be rang'd into the form of a Square (7. 13). Let them then be diſpoſed into the Square *a c*. So ſhall the Square *a c* be equal to the Plane A C. for 'tis only the ſame Number diſpos'd or rang'd after another manner. Wherefore (6. 59) the ſide *a b* 4 ſhall be a mean Proportional between A D 2. and A B 8.



29. Between two Numbers Non-ſimilar a mean Proportional can't be found. Let the Numbers be 4. and 6. Range each of them into a Right-line, and multiply them, they will produce the Plane 24. But this Plane 24 is not a ſquare Number (7. 25). and conſequently cannot be rang'd into a ſquare Form. Wherefore 'tis impoſſible to have any *Mean* between 4. and 6. For ſuch a pretended mean Proportional muſt, multiplied by it ſelf, produce a Square, which (as hath been prov'd elſewhere) will be equal to the Plane made between 4. and 6. (6. 59). which is impoſſible. becauſe this Plane 24, made out of 4 and 6. is not a ſquare Number.

30. Let there be two Lines *a e* and *e c*, ſo to one another, as one Number to another Nonſimilar. *v. gr.* as 1 to 2. Let alſo *e b* be a mean Proportional, ſo that *a e. eb* : : *e b. e c.* I ſay, that *eb* is *Incommenſurable* with the two Extremes *a e* and *e c*. For *a e* and *e c* being as 1. to 2. (*i. e.*) as Numbers Non-ſimilar (by the Suppoſition) as alſo are all their Equimultiples (7. 27). 'tis impoſſible to find a mean Proportional between *a e* and *e c* (by the Precedent) and conſequently, *e b* cannot be to *a e*, or to *e c*, as Number to Number, Wherefore it is Incommenſurable with them,



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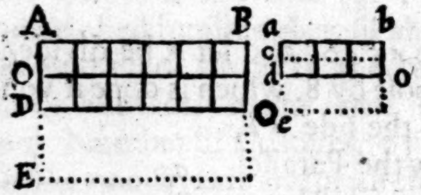
26. If two Numbers are ſimilar Planes, their *Equi-multiples* and any of their (*Reſpectively*) equal Parts, are alſo Similar Planes. Let the Planes be *a b c d*. 3. and AB CD. 12. ſo that *a b*. A B ; : *b c*. B C. I ſay if you take the double of the one and the double of the other (or any other Equi-multiple, be it what you pleaſe) thoſe doubles ſhall be Similar.



For having taken *a e* double to *a d*, and A E double to A D: in order to make the Plane *b e* double to *b d*, and B E double to B D: 'Tis clear that *a d*. A D : : *a e*. A E. But *a d*. A D : : *a b*. A B. wherefore alſo *a e*. A E : : *a b*. A B. And conſequently the Planes *b e* and B E are Similar.

'Twould be the ſame thing had you taken their halves *b o* and B O,or any other equal Parts of each.

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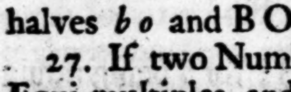
28. Between any two ſimilar plane Numbers whatſoever, there is to be found a mean Proportional. Let the two Nnmbers be 2 and 8, I ſay it is poſſible to find a Number which ſhall be a mean Proportional between them. For if we imagine the Plane 8 to be ranged in a right line A B, and the Plane 2, alſo to be ranged in another Right-line as A D. and that out of thoſe two Right-Lines there be formed the Plane A C. 16. That Plane A C. 16. will be produced by the Multiplication of the two Numbers 2 and 8 (6. 17 and the following Propoſitions) and conſequently the Number of the lit-



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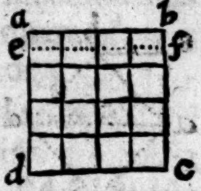

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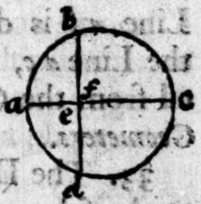
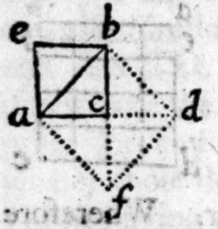


31. The Diameter of a Square *a b* is Incommenſurable to the ſide *a c.* For taking *a d* double to *a c*, and making the Triangle *a b d*, it ſhall be Similar to the Triangle *a b c* ; becauſe *c d* being equal to *c b*, the Angle *d* is equal to *c b d* (2. 15). and the Angle *d* muſt be a Semi-right one as well as *c a b* ; wherefore *a b d* is a Right-angle ; and conſequently *a c. a b : : a b. a d.* That is, *a b.* is a mean Proportional betwenn *a c* 1. and *a d* 2. and therefore Incommenſurable (by the Precedent).



COROLLARY.

Hence 'tis impoſſible to Expreſs one Square that ſhall be Double of another in Rational Numbers.

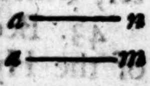
32. The *Power* of a Line is the Square, which is made upon it. Thus the Power of the Line *a c* (*Fig. preced.*) is the Square *a e b c* ; and the Power of the Line *a b* is the Square *a b d f.* And we ſay that Line *a b* is double in Power (in Latin *bis poteſt*) to the Line *a c*, which is a manner of Speaking borrowed from the *Greeks*, and generally received amongſt *Geometers.*

33. The Diameter *a b* is *Commenſurable in Power* to the ſide *a c* : That is, it's Square *a b d f* is Commenſurable to the Square *a e b c* , for 'tis indeed double to it.

34. But if you take *a o*, a mean Proportional between *a b* and *a c*, that mean *a o* ſhall be Incommenſurable to them even in Power; *i. e.* the Square of *a o* is Incommenſurable to the Square of *a b*, or to the Square of *a c*. for the Square of *a c* to the Square of *a b*, is in a Duplicate *Ratio* of *a c* to *a o* (6. 22); that is, as *a c* to *a b* (6. 30). But *a c* is Incommenſurable to *a b* (7. 31.) wherefore the Square of *a c* is Incommenſurable to the Square of *a o*.

35. There is a *Second Power* of a Line which is call'd the Cube, which is made by multiplying the Square, by that firſt Line, or Root.

36. If two mean Proportionals *a n* and *a m* be taken between *a c* and *a b*; ſo that *a c*. *a n* :: *a m*. *a b* the Line *a n* will be Incommenſurable in this ſecond Power to *a c* (*i. e.*) The Cube of *a c* will be Incommenſurable to the Cube of *a n*, becauſe the Cube of *a c* to the Cube of *a n* is in a *Triplicate Ratio* of the ſide *a c* to the ſide *a n*; *i. e.* as *a c* to *a b*. But *a c* and *a b* are Incommenſurable, wherefore, &c. However *a c* and *a b* are Commenſurable in the ſecond Power, for the Cube of *a b* is double to the Cube of *a c*.



37. 'Tis eaſie to apply to *Solid Numbers* what hath here been ſaid of Plane ones. And thoſe are call'd *Solid Numbers* which ariſe from the Multiplication of a Plane *Number* by any other whatſoever. *v. gr.* 18 is a ſolid Number made of 6 (which is a plane) multiplied by 3; or of 9 multiplied by 2.

38. *Similar Solid* Numbers are thoſe, whoſe little Cubes may be ſo rang'd as to make ſimilar and rectangular Parallelopipeds.

39. *Cubick Numbers* are ſuch as can be rang'd into the form of Cubes as 8. or 27, whoſe *ſides* are 2 and 3 and their *Baſes* 4 and 9.

40. Every Cubick Number multiplying another Cubick Number produces a third Cubick Number.

41. Between two ſimilar ſolid Numbers there may be found two mean Proporionals.

That which hath been demonſtrated in reſpect to Plane Numbers, may be applied to Solids.

42. Theſe Demonſtrations by which 'tis prov'd that there are Incommenſurable Lines and Magnituds, ſhew alſo that a *Continuum* is not compos'd of Finite Points: For if the Diameter as well as the ſide of a Square were compos'd of Finite Points, a Point would meaſure both the ſide and the Diameter: for that Point would be found a certain Number of Times in the ſide & another determinate Number of times in the Diameter, which the preceding Propoſitions prove impoſſible.

43. Becauſe in a Rectangle Triangle the Square of the Hypothenuſe is equal to the Sum of the Squares of the Legs; (6. 61). we have alwas uſed this Triangle for the diſcovery of Incommenſurables. For if all the three ſides are Commenſurable, they may be all three expreſs'd by three Numbers, and then the Square of the greateſt Number will be equal to the Sum of the Squares of the other two. As if the greateſt ſide or Hypothenuſe be 5 Feet, the leaſt ſide three, and the middle one 4: The Square of 5 will be 25, the Square of 3, 9; and the Square of 4 will be 16: And 9 and 16 added together do make the great Square 25. But if the leaſt ſide of ſuch a Triangle be 2. and the middle one 3. then the greateſt ſide cannot be expreſs'd in Numbers, becauſe the Square of the leaſt ſide 4 added to the Square of the middle ſide 9 makes 13, which expreſs the Square of the greateſt ſide. But as that Number 13 is not a ſquare Number, ſo its ſide or Root cannot be expreſs'd by any Number.

44. At all times Men have been Sollicitous to find out ſome Method of diſcovering proper Numbers to expreſs the three ſides of a Rectangle Triangle, ſo as

to be aſſured that all the three ſides are Commenſurable. Therefore I here ſhew you ſuch a Method, by which you may find out all the poſſible Numbers that are proper for this purpoſe.

45. If you take any two Numbers (even Unity it ſelf) differing but by an Unite, and add the Squares of them together, the Sum will be a Number which ſhall be the Root of a Square equal to two Squares; And that number will expreſs the greateſt ſide of a Rectangle Triangle, whoſe middle ſide ſhall be that Number leſſen'd by Unity, and the leaſt ſide ſhall be the Sum of the two firſt Numbers. *v. gr.* Having taken 1 and 2 and Squared each of them you have 1 and 4; Add thoſe two Squares together, and the Sum is 5 I ſay 5 will expreſs the greateſt ſide; and then 4 will be the middle one and 3 the leaſt; and 25 the Square of the Hypothenuſe will be equal to the Sum of the other two Squares. In like manner if you take 2 and 3, and add the Squares 4 and 9 together, the Sum is 13. Then I ſay will 13, 12 and 5 be three ſides of a Rectangle Triangle; ſo that 169 the Square of 13, ſhall be equal to 144 and 25, the Squares of 12 and 5. Moreover if you take 3 and 4: the Sum of their Squares 9 and 16, makes 25; wherefore I ſay 25 may be the greateſt ſide of a Rectangle Triangle, whereof 24 will be the middle ſide, and 7 the leaſt ſide.

It muſt be obſerv'd alſo, that the Equimultiplies of any 3 Numbers thus found will do the ſame thing: Thus, having found 5, 4 and 3 their doubles 10. 8. and 6. will repreſent the three ſides of a Rectangle Triangle, ſo that 100 the Square of 10 ſhall be equal to the Sum of 64 and 36 the two Squares of 8 and 6. And their Triples alſo 15. 12. and 9. will do the ſame thing: For any one may ſee that all theſe Numbers ſtill having the ſame Proportion, do as it were conſtitute but one only Triangle, *viz.* that which is expreſs'd by 5. 4. and 3. And therefore all thoſe Numbers may be taken for the ſame.

N. B. *The three Sides of a Rectangle Triangle will then only be Commenſurable, when they are in this Proportion,* viz. *as* $aa+ee$, $aa-ee$ *and* $2ae$. *That is, the Sum of two Square Numbers, the difference of their Squares, and the double Rectangle of their Roots.*

ELEMENTS OF GEOMETRY.

BOOK V.

Of Progreſſions *and* Logarithms.

1. *Progreſſion* is a *Series* or Rank of Quantities which keep between one another any kind of ſimilar Relation or Proportion; and every one of theſe Quantities is called a *Term*.

2. When the Terms which ſo follow one another do equally Increaſe or Decreaſe, the Progreſſion is called *Arithmetical*; as are all Numbers proceeding according to the natural Order of the Figures, as 1, 2, 3, 4, 5, 6, *&c.* As alſo all odd Numbers, as 1, 3, 5, 7, 9, 11, &c. or as 4. 8. 12. 16. or as 20. 15. 10. and the like.

3. Arithmetical Progreſſion may be increaſed Infinitely, but not diminiſhed.

4. If in an Arithmetical Progreſſion there be four Terms, the Difference between the two firſt of which is equal to the difference between the other two, thoſe four Terms are ſaid to be *Arithmetically Proportional:* As in the Progreſſion of the natural Numbers 1, 2, 3. 4, 5, 6, 7, 8, 9, &c. If you take four as 2. 3 : : : 9. 10 (*This mark* : : : *I ſhall for the future uſe to ſignifie Arithmetical Proportion*) there will be the ſame Arithmetical Proportion between 2 and 3 as there is between 9 and 10; that is, 10 exceeds 9, as much as 3 doth 2: ſo alſo are 3. 5. : : : 8. 10. are in Arithmetical Proportion; and ſo are 1. 5 : : : 5. 9. where 5 being taken twice, is in Arithmetical mean Proportional between 1 and 9.

5. In Arithmetical Proportion the Aggregate or Sum of the two extreams is equal to the Aggregate of the two Means, as in 2. 3 : : : 9. 10. the Sum of 2 and 10 is equal to the Sum of 3 and 9, that is 12; ſo alſo in 3. 5 : : : 8. 10. The Aggregate of 3 and 10 is 13, which is equal to the Aggregate of 5 and 8. And the reaſon of this is ſelf-evident. For tho' 10 exceeds 8, yet that which is added to 8 (*viz.* 5) doth juſt as much exceed 3, which is added to 10, and ſo there neceſſarily ariſes an Equality between them.

6. The Sum of the firſt and laſt Terms in any Arithmetical Proportion is equal to the Sum of the ſecond and the laſt ſave one; or to the Sum of the *third from the firſt Term, added to the third, accounted backward from the laſt,* &c. as in the firſt Example, 1 and 9 make 10; and ſo do 2 and 8, 3 and 7, or 6 and 4 always make 10. And in the middle remains 5, which being taken twice (as if it were equivalent to two Terms, becauſe 'tis equally diſtant from the firſt and laſt Term) makes alſo 10.

7. If you add the firſt Term to the laſt, and multiply that Sum by half the Number of the Terms, the Product ſhall be equal to the Aggregate or Sum

of all the Terms. As in the former Example, 1 added to 9, makes 10, and 10 multiplied by 4$\frac{1}{2}$ (or 4, 5) for there are 9 Terms, produces 45 which is the Sum of all the Terms from 1 to 9. As is manifeſt from the Precedent.

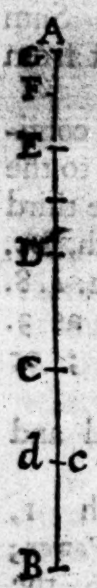
8. When the Terms of the Progreſſion are continual Proportionals; that is, when the firſt is to the ſecond as that is to the third Term as the third is to the fourth, and as the fourth is to the fifth, *&c.* then the Progreſſion is call'd *Geometrical* as 1. 2. 4. 8. 16: 32 ∷. Or as 1. 3. 9. 27. 81 ∷ or again, as 3. 12: 48. 192. 768. or *deſcending*, as 8. 4. 2. 1 ∷; or laſtly as $\frac{1}{2}$. $\frac{1}{4}$ $\frac{1}{16}$. $\frac{1}{32}$ $\frac{1}{64}$, *&c.*

9. Geometrical Progreſſion may be encreas'd and diminiſh'd Infinitely.

10. When the Progreſſion begins with 1, the ſecond Term is call'd the *Root, Side* or firſt *Power*; the *third* is call'd the *Square* or ſecond *Power*; the *fourth*, the *Cube* or third *Power*; the *fifth*, the *Biquadrate* or *fourth* Power; the *ſixth*, the *Sur-ſolid* or *fifth Power*; the *ſeventh* the *Quadrato Cube* or *ſixth Power*, &c

11. If (*in ſuch a Progreſſion*) you take four Terms, the former two of which are as much diſtant from each other, as the two latter are: Thoſe are ſimply Proportional, and the Rectangle of their extreams is equal to that of their two middle Terms.

12. Let the Quantity A B be ſo divided in C, D, E and F, *&c.* that A B. A C : : A C. A D : : A D. A E, *&c.* Then I ſay B C. C D. D E. E F, *&c.* are in continual Geometrical Proportion; and alſo that A B. A C : : B C. C D : : C D. D E, *&c.* for becauſe A B. A C : : A C. A D. it will follow by Diviſion of Proportion, that A B. leſs A C. (that is C B.) A C : : as A C leſs A D. (that is D C) A D. and conſequently Alternately C B. C D : : A C. A D or as A B. A C. and ſo of all others it may be proved : : D C. E D : : E F : : G F, *&c.*



13. Let there be a Progreſſion of Quantities in a Right-line B C, C D, D E, E F, *&c.* let C *d* be equal to the *ſecond* Term C D, that ſo we may have B *d* the difference between the *firſt* and *ſecond* Terms: And let it be made as B *d*. B C : : B C. to a fourth Line, *viz.* B A. I ſay, that if the Number of the Terms B C. C D. D E, *&c.* be Finite, tho' never ſo great, all thoſe Terms taken together, although there be an hundred thouſand Millions of them, ſhall be leſs then B A. But if we ſuppoſe the Progreſſion infinite, or that the Terms are infinitely many, then ſhall all of them taken together be exactly equal to B A. For ſince by the ſuppoſition B *d*. (that is B C leſs C *d* or C D) is to B C : : B C. (that is A B leſs A C) A B. it may eaſily be found that as B C. C D : : A B. A C : : A C. A D, *&c.* and conſequently all the Terms C D, D E, E F, *&c.* will always be found within, or be hither the Point A. To which it Approaches the nearer, the more the Number of the Terms is increas'd. So that we ſee plainly, that all theſe Terms (which in Books are uſually call'd *Parts Proportional*) tho' they be actually Infinite, cannot make an Infinite length, becauſe they will all be included within the Line B A.

14. This Demonſtration will appear much more eaſie and ſenſible by the Example of a particular Progreſſion, where the Terms are in a double *Ratio v. gr.* Let C B be double to D C and D C double to D E, *&c.* For if the Number of the Terms be here Finite tho' it be an hundred thouſand Millions, and you take the laſt and leaſt Term, for Example F E, and add to it another Quantity, as ſuppoſe A F, equal to it: It is then plain that E A muſt be equal to the Term E D, which is the laſt ſave one; For E D is double to E F by the Suppoſition (the *Ratio*

being every where double) and E A is alſo double to E F by the Conſtruction, it having been made ſo, by taking F A equal to E F. In like manner A E with D E, that is A D, ſhall be equal to the following Term C D, and at laſt A C will be equal to B C. So that from hence it appears, that the firſt or greateſt Term is always equal to all the others taken together, provided there be added to them but a Quantity equal to the laſt and leaſt Term; but if nothing be added, the firſt Term is always greater than the Sum of them all.

If theſe Terms are ſuppos'd to be actually Infinite, then the greateſt B C will be exactly equal to all thoſe Infinite others taken together C D, D E, E F, *&c.* For any one may eaſily diſcern, that the more there are of ſuch Terms, the more you approach towards A: by cutting off ſtill the half of the remainder: But when any Quantity is thus leſſen'd by half, and the Remainder again by half, and then the half of that third Remainder taken and ſo on: 'Tis plain, that by ſuppoſing the Diminution to be made an Infinite number of times, nothing at laſt will Remain.

This alſo might be demonſtrated by a *Reduction ad Impoſſible*, by ſhewing that all thoſe infinite Terms are, taken together, neither greater nor leſs than A B.

15. Hence may the Difficulties raiſed by the Schoolmen againſt the (*Infinite*) Diviſibility of a *Continuum* be ſolved, tho' to Perſons Ignorant of *Geometry* they appear Unſolvable: But indeed at the Bottom, they are nothing but meer Paralogiſms.

16. If two Progreſſions are ſuppoſed, one Geometrical beginning with 1. and the other Arithmetical beginning with 0, ſo that the Terms in one ſhall be placed over and anſwer Reſpectively to thoſe in the

other; The Arithmetical ones are call'd *Logarithms*, *Exponents*, (or *Indexes*) as in the following Ranks.

0. 1. 2. 3. 4. 5. 6. 7. 8.
1. 2. 4. 8. 16. 32. 64. 128. 256.

17. That which is produced in a Geometrical Progreſſion by Multiplication and Diviſion, is effected in the *Logarithms* by Addition and Subſtraction: As, if having three Numbers given; 2. 8 : : 64; You would find a fourth Proportional to them in Geometrical Progreſſion: You muſt multiply 64. by 8. (which are the two middle Terms) For the Product, 512. ſhall be equal to the Product made by 2. and the fouth Number ſought, they being the two extreams of four Proportionals. And to find this fourth Number, you need only divide 512 by 2, and the Quotient will be 256. So that 2. 8 : : 64. 256. and 64 and 256. will be juſt as far diſtant from one another in the order of the Progreſſion, as 2. and 8 are.

But if inſtead of the Geometrical Numbers 2. 8 : : 64. you had uſed their Logarithms 1. 3 : : 6, which Anſwer to them in the Progreſſion, and were minded to find a fourth Logarithm, then you muſt have added 3 and 6 which make 9, and from thence have Subſtracted 1, there would remain 8. The Logarithm anſwering to the Geometrick Number 256.

18 So alſo, if there be two Geometrick Numbers 4 and 8, to which the Logarithms 2 and 3 do anſwer; by multiplying 4 by 8 you produce 32; the Number under the Logarithm 5, which is the Sum of the Logarithms of 2 and 3.

19 In like manner by multiplying 16 by it ſelf, there will be produced 256 which ſtands under the Logarithm of 8, the Sum of 4 added to it ſelf.

20. So if the Geometrical Number were required that ſhall anſwer to or ſtand under the Logarithm 16; you muſt take 256 which ſtands under 8, and multiply it by it ſelf, and it will produce 65536 the Number requir'd.

21. If moreover the Geometrical Number anſwering to the Logarithm 23 were required, you may take any two Logarithms whoſe Sum is 23, as ſuppoſe 7 and 16 and multiplying the Geometrical Number under them, *viz.* 128 and 65536 one by another, the Product will be 8388608. The Number which ought to ſtand under the Logarithm 23, or in the 23 place of a Series of Geometrical Proportionals beginning from 1.

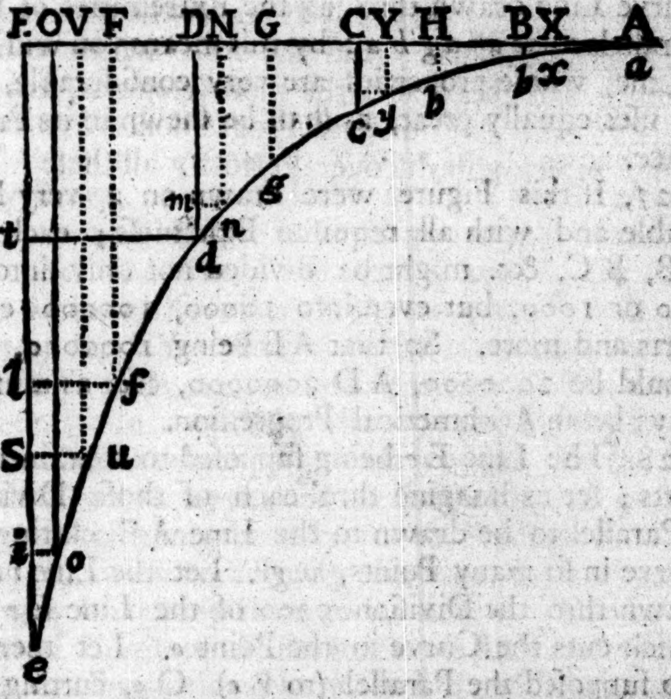
22. From hence appears the way of Anſwering that ordinary Queſtion, how much a Horſe would coſt, if bought on this Condition: That for the firſt Nail in his Shoe a Farthing were to be paid, for the ſecond Nail two Farthings, for the third Nail four Farthings, for the fourth Nail eight Farthings, and ſo on, ſtill doubling for 24 times: For the 23d place in ſuch a Progreſſion would be the laſt Number 8388608 Farthings, which being reduced is 8738 *l.* 2 *s.* 8 *d.* and being doubled according to (8. 14.) gives the whole price of the Horſe 17476 *l.* 5 *s* 4 *d.*

23. Where two compleat Progreſſions are fitted ſo as to anſwer one to another, the Geometrical to the Arithmetical; as ſuppoſe in Tables for that purpoſe calculated in Books, there abundance of Pains and Labour is ſpared, in finding the Geometrical Numbers: For Inſtance, let thoſe three Numbers 32. 64. 128. be given, and that a fourth Proportional were required: Inſtead of multiplying 64 by 128, and dividing the Product by 32 (which way is very tedious in great Numbers): you need only take the Logarithms of the three given Numbers, *viz.* 32. 64.

128; and adding the 2d and 3d together, from their Summ, Subſtract the firſt, the difference will be the Logarithm of the correſponding Geometrick Number 256.

24. But becauſe in ſuch a Geometrical Progreſſion all Numbers will not be found, this Medium hath been diſcovered; they have calculated two Progreſſions, one of which, contains all Numbers 1. 2. 3. 4. 5. 6. 7. 8, *&c.* which ſeems to be an Arithmetical Progreſſion, but yet hath in reality the Properties of a Geometrical one. And the other which contains Numbers in appearance the moſt Irregular, is nevertheleſs a true Arithmetical Progreſſion. See here a Line, which will diſcover Perfectly all theſe Myſteries.

25. Let the Right-line A E be divided into the equal Parts A B, B C, C D, D E, *&c.* from the Points A, B, C, D, E, *&c.* let the Lines A *a*, B *b*, C *c*, D *d*, and E *e*, be drawn all (*Perpendicular to* A E *and conſequently*) Parallel to one another: And let them be all in a Geometrical Progreſſion: As, let A *a* be 1, B *b* 10, C *c* 100, D *d* 1000,



E *e* 10000, *&c.* Then ſhall we have two Progreſſions of Lines, the one Arithmetical and the other Geometrical: For the Lines A B, A C, A D, A E, are in Arithmetical Progreſſion, or as 1. 2. 3. 4. 5, *&c.* and ſo do repreſent the Logarithms; to which the Geometrical Lines A *a*, B *b*, C *c*, *&c.* do correſpond.

26. Let each of the equal Parts E D, D C, C B, &c. be divided equally again in F. G. H, and let the Parallels F *ſ*, G *g*, &c. be drawn, and be mean Proportionals between the Collateral ones; that is, *E e*: F*f*: : F*f* D*d*: : D *d*. Gg, Let there alſo be more mean Proportionals drawn from the middle of each Subdiviſion E F, F D, D G, and ſo on, till theſe Parallel Lines growing very numerous, have at laſt but a very ſmall diſtance from each other; then imagine a Curve Line drawn thro' all the Extremities of theſe Parallels as *e o u f d g h a*: by this means you will gain a Line, whoſe properties are very conſiderable, and its uſes equally great, as ſhall be ſhewn in its Proper Place.

27, If this Figure were drawn on a very large Table and with all requiſite Exactneſs; each part A B, B C, &c. might be divided not only into an 100 or 1000, but even into 10000, 100000 equal Parts and more. So that A B being 1000000, A C would be 2000000, A D 3000000, &c. as muſt always be an Arithmetical Progreſſion.

28. The Line E *e* being ſuppoſed to contain 1000 Parts; let us imagine thro' each of thoſe Diviſions a Parallel to be drawn to the Line A E cutting the Curve in ſo many Points, *v. gr.* Let the Line *i o* be drawn thro' the Diviſion 9900 of the Line E *e* and which cuts the Curve in the Point *o*. Let there be alſo ſuppoſed the Parallel (to E *e*) O *o*, cutting the Line A E in the Diviſion 399563. Then any one may know that 399563 is the Logarithm of the Number 90000. In like manner if S *u* paſſed thro' the Diviſion 9000 of the Line E *e*, and the Line *u v* were drawn cutting A E in 395424, then would that Line *u v* be the Logarithm of 9000, &c.

29. So that by this means a Table of Logarithms from 1 to 10, 000 may eaſily be made; and farther, by producing the Line A E.

30. Note, to obtain all the Logarithms from 1 to 10000; 'twill be enough to ſeek the Logarithms from 1000 to 10000: That is (having drawn the Parallel *d t*) to take the Logarithms of all the Diviſions from *t* to *e*, which Logarithms are all contained between E and D. For by this you will have the Logarithms of all the Parts that are between *t* and *E*; and whoſe Logarithms lie between D and A: For Example, ſince O *o* is 9900 Parts, and its Logarithm 399563, the ſame Number may be taken for the Logarithm of 990 which is N *n*; as alſo of the Number Y *y* 99, changing only the firſt Figure 3. Becauſe according to the compoſition of this Line O N or N Y ought to be equal to E D or D C, as one may eaſily Prove. So that O N or N Y will contain 100, 000 and becauſe A O is 399563, ſubſtracting O N 100, 000, there will remain 299563 for A N, from whence alſo taking 100, 000, there will reſt 199563 for A Y. And after the ſame manner, having A Y 3995424 for the Logarithm of V *u* which is 9000; you may have alſo 095424 for the Logarithm of X *x* which is 9. Or 195424 for the Lognrith of 90, or 29524 for the Logarithm of 900.

31. All this may be reduced to Practice for Calculation, without actually drawing theſe Figures, but only Imagining them to be drawn. For by the Rules of Common Arithmetick we may find out F *f*, the mean Proportional between D *d* and E *e*, and after that, another Mean between D *d* and F *f*, or between F *f* and *E e*, &c. But what we have here explained is ſufficient to gain as much Knowledge as is neceſſary for us to have of the Nature and Compoſition of Logarithms: There being no need for us to undergo the Labour of Calculating Tables of Logarithms; ſince it's already ſo well and ſo often done to our Hands. God, for the Publick Good, having raiſed ſome Perſons, whom he was pleaſed to endow with ſufficient Patience to ſurmount ſo tedious and

laborious a Work, as one would think to be inſuperable. For we know that above 20 Men were engaged in ſuch a Calculation, for above 20 Years together, with indefatigable Induſtry and Aſſiduity.

(Pardie *ſpeaks here a little Covertly, ſeeming willing to Inſinuate that this moſt uſeful and admirable Work was done firſt in his own Country, whereas the Logarithms were the Invention of my Lord* Neper *a* Scotch *Baron, and the firſt* Tables *were calculated by him, with the Aſſiſtance of our Countryman Mr.* Henry Briggs).

Of late ſeveral Improvements have been made in this Matter: As by *Nicholas Mercator,* of which ſee Dr. *Wallis*'s Thoughts in *Philoſoph. Tranſact.* 38. John Gregory *hath given us a way to find Logarithms to* 25 *Places by help of the Hyperbola. But Captain* Halley *in* Philo. Tranſact., N°. 216. *ſhews a way from the bare Conſideration of Numbers, and withal by the help of Mr.* Newton'*s Way to find the* Unciæ *of the Numbers of a Binomial* Power, &c. *By which you may find compendiouſly the Logarithms of all Numbers to above* 30 *places. And he gives there ſeveral ſeries for this purpoſe, ſome Univerſal, and ſome appropriated to a peculiar ſort of Logarithms.*

ELE-

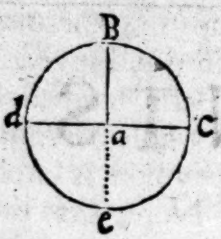
ELEMENTS OF GEOMETRY.

BOOK IX.

Problems *or* Practical Geometry.

THAT Propoſition is called a *Problem* in (Geometry) which teaches us how to do any thing, and demonſtrates alſo the Practice of it: Whereas *Theorems* are Speculative Propoſitions, in which are conſidered the Affections and Properties of Things *already done.*

2 *To Divide a Circle into four and into 6, and all Arks into two equal Parts.* To divide it into four Parts draw two Lines as *d a c* and B *a c* at Right-Angles to each other. To divide it into eight Parts; Biſſect the four Arks B *c*. *c e*, &c. which is done by ſtriking (without the Ark B *c*) two other Arks, with the ſame opening from the Points B and C, for if a Line be drawn, from the Point where thoſe Arks croſs each other, to the Center *a*, it ſhall Biſſect the Ark B C. The like is to be done for the other Arks.

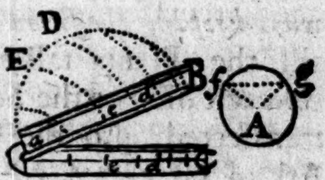


To divide a Circle into ſix equal Parts; you need only take the Length of the Radius: and applying it ſix times about the Circle, it will eactly divide the Circumference into ſix equal Parts, and thus by a new Biſſection, may a Circle be divided into 12, 24, 48, or into any Number of equal Parts, &c.

3. *To divide a Circle into five, into fifteen, and into other Equal Parts.* This may be done thus; (as I demonſtrate in *Algebra*). Make a Rectangle Triangle, one of whoſe Legs ſhall be the Radius of the Circle, and the other half the Radius. From the Hypothenuſe of this Triangle, take half the Radius the remainder ſhall be the Chord of 36 *deg.* and the ſide of a Decagon. Double that Ark, you have the Ark of 72 *deg.* (*whoſe Chord is the ſide of a Pentagon*) and it is the fifth part of the Circumference; and the ſaid Chord ſhall be alſo the Hypothenuſe of a Rectangle Triangle, one of whoſe ſides is the Radius and the other the ſide of Decagon. And as by the laſt was found the Chord of 60 *deg.* ſo by ſubſtracting the Chord of 36 *deg.* from 60 *deg.* youmay have the Chordof 24*deg.* which is the 15th part of the Circumference. But for Practice, the ſhorteſt and ſureſt way, is by repeated Trials with the Compaſſes to find a

Diſtance that will go preciſely five times about the Circle: Then divide, after the ſame manner (by Tryals) that diſtance into three equal parts exactly So ſhall you gain a Chord that will divide the Circumference into 15 equal Parts, and then dividing each of thoſe 15 Chords into four equal Parts, and each of thoſe into ſix; you will divide the whole Circumference into 360 *deg*. And this Diviſion is moſt Commodious for Practice and Uſe. Note, that the way to divide a Circle into 3, 5, 7, or into any other odd Number of Parts is not yet found Geometrically; Geometrically I ſay, that is, by making Uſe only of a ſtrait Line and a Circle.

“ This Diviſion of a Circle into 360 *deg*. is very
“ uſefull, when a Perſon underſtands how to uſe the
“ *Compaſſes of Proportion* (or
“ *Sector*.) 'Tis ſo called be-
“ cauſe 'tis a kind of Com-
“ paſſes with broad Legs:
“ As *a* B. *a* C. on which are
“ deſcribed divers Lines and
“ Diviſions but thoſe which
“ are moſt in uſe, are of two ſorts. On one ſide of this
“ Sector, and on each Leg, is a Line *a e* B and
“ *a e* C, which ſerves to divide a Circle into 360 *deg*.
“ at once, and alſo to take at any time as many De-
“ grees as you pleaſe. And this Line on the Sector
“ is thus divided.

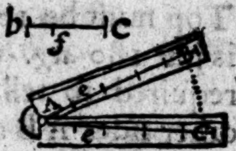


4. *To divide and graduate the Sector, that it may ſerve for the Diviſion of a Circle.* Imagine a Semicircle *a* E D B accurately divided into 180 *deg*. if then from the Point *a*, as from a Center, you transfer the Diviſions of the Semicircle into a Line *a* B. *v. gr.* If from E, 60 *deg*. you draw the Ark E *e*, and if from D 90 *deg*. in the Semicircle, you draw the Ark D *d*, &c. Then ought 60 *deg*. on that Leg of the Sector, to be placed at the Point *e*, and 90

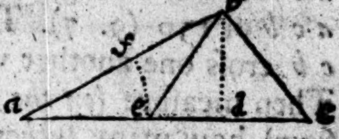
deg. at the Point *d*, &c. And if you Transfer the ſame Degrees after the ſame manner into the other Leg *a* C, you will graduate the Lines *a* B and *a* C (on the Sector) as they ought to be for this Purpoſe, *and they will be two ſimilar Lines of Chords.*

5. *To explain the Uſe of the Sector as far as it ſerves for the Diviſion of a Circle.* Let there be a Circle given A *f*: take with your Compaſſes the Radius A *f* and (*keeping that Diſtance*) ſet one Foot of them in *e* or 60 *deg.* on one Leg of the Sector; move the other Leg of the Sector to and fro ſo long, till the other Point of the Compaſſes falls exactly on *e* or 60 *deg.* in that Leg of the Sector: So that the Diſtance *e e* be exactly equal to the Radius A *f*. Then if you would have readily 90 *degrees* of that Circle; (*Letting the Sector lye ſtill, and always keeping the ſame Angle*). Open your Compaſſes till the Points fall exactly on *d* and *d* or 90 *deg.* on each Side of the Sector: And then that Diſtance tranſferred into the Circle, in *f*, *g*, gives you the Ark of 90 *deg. f. g.* So alſo if you would have had 35 *deg.* you need only apply your Compaſſes to 35 *deg.* and 35 *deg.* on each Leg of the Sector in the Lines (*of Chords*) *a* B and *a* C; and that diſtance tranſferred into the Circle, ſhall cut off the Ark of 35 *deg.* and thus may you proceed to find any Degrees you pleaſe. All which is grounded on the 42, 43, 49. and 50 Propoſitions of the VI Book. For ſince all Circles are ſimilar Figures, (6 50) the Chord *f g* will be to the Radius *f* A : : as the Chord *d d* to the Radius *e e*; that is, as *a d* is to *d e*. Now 'tis plain from what hath been proved elſewhere, that the Triangles *a d d* and *a e e* are ſimilar; and therefore *d d*, *e e* : : *a d*. *a e*. But *d d* is by the Conſtruction equal to *f g*. and *e e* to A *f*. wherefore *f g*. A *f* : : *d a*. *a e*. Q. E. D.

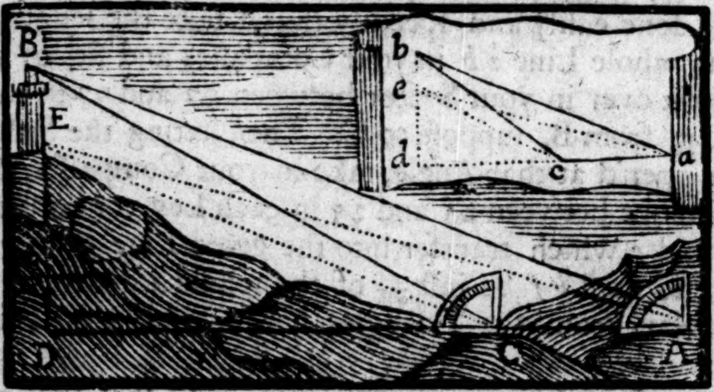
6. *To divide the Line of* Equal Parts *or* Lines *on the Sector, for the Diviſion of any Right-lines given.* There being two Right-lines drawn from the Center of the Sector on the Legs as *a* B and *a* C: Let each be divided into 100 or 200 equal Parts: And then they will ſerve to divide any Line given into any Number of equal Parts; As for Inſtance, let the Line given be *c b*, and that you were required to take $\frac{25}{97}$ parts of it. Now to divide the whole Line *c b* into 97 equal Parts, and then to take 25 of them according to the common way of dividing Lines, would be very tedious: But by the Sector 'tis done eaſily and ſpeedily thus, take the length of the whole Line *c b* in your Compaſſes and fit, or apply it over in your Sector between 97 and 97 in each Leg, from B, ſuppoſe to C. Then letting the Sector lie open'd at that Angle, take in your Compaſſes, the diſtance between 25 and 25 in each Leg, or between *e* and *e* which transfer into the given Line from *b* to *f*; ſo ſhall *b f* be juſt $\frac{25}{97}$ of the whole Line *c b*: As is plain from the Triangles A B C and A *e e* being Similar.



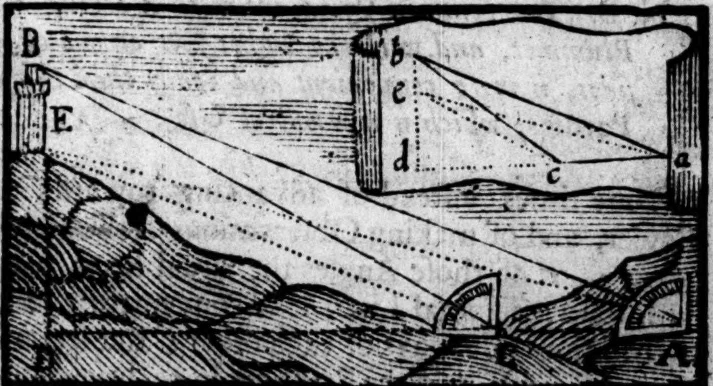
7. *On a Line given to make an Angle that ſhall contain any Number of Degrees aſſign'd.* Let the Line given be *a c*, on which 'tis required to make an Angle of 30 *deg.* From the Point *a* as from a Center ſtrike the Ark *f e*, from which take by the Sector, or otherwiſe, 30 *deg.* from *e* to *f*; then thro' *f* draw the Line *a f*, which with the Line *a c* will make an Angle of 30 *deg.*



8. *Having the Angles of any Triangle and one ſide given to find the other two ſides.* Suppoſe you are told there is a certain Triangle ſome where, whoſe Baſe A C is 10 Fathom; and that the two Angles at the Baſe are A C B. 150 *deg.* and C A B 20 *deg.* (and conſequently the remaining Angle at the Vertex or Top muſt be 10 *deg.* for the Sum of 150, 20 and 10, is juſt 180 *deg.* which is two Right Angles). You are required to tell how many Fathom there are in the other ſides A B and AC. Make on Paper, or rather on fine Paſtboard, a Triangle *a b c* Similar to the propos'd one after this manner. Take a Baſe at pleaſure *a c*, and from any Scale of equal Parts let it be 10 Inches half Inches, &c. in Length. On this Line *a c* make two Angles, one *cab* of 20 *deg.* and the other *a c b* of 150 (9. 7). Then will the two Lines *a c* and *c b* croſs one another when produced in the Point *b*. Then meaſure (*on the ſame Scale you took the Baſe* a c *from*) how many *Inches*, &c. the Lines *a b* and *c b* are in Length; And you may be aſſured that there are juſt ſo many Fathom in the Lines A B and C B ſought, as you find Inches, &c. or any equal Parts, in the Lines *a b* and *c b*. For ſince the Triangles are Equiangular, they are Similar, and therefore *a c. a b* :: A C, A B, &c.



9. *To meaſure Diſtances, Heights, Depths, and in general the Dimenſions and Magnitudes of all Remote and Inacceſſible Places.* If on the top of any Hill appearing at a diſtance there were a Tower as B E and its diſtance from us and its Height were required: You muſt firſt with ſome Inſtrument (as with a *Quadrant*, that is the fourth part of a Circle divided into 90 *deg.* and furniſh'd with a Ruler, or Label with Sights, and moveable on the Center) you muſt I ſay with ſome ſuch Inſtrument, take two Angles at two ſeveral Stations in this manner: If you are in the Station A, place your Inſtrument ſo, that one ſide of it may anſwer exactly to the Horizontal Line A D; and keep it without raiſing or depreſſing it in this Poſition. Then place your Eye at A, (that is at the Center of the Inſtrument), and turn the Label till it Point to the Top of the Tower B, and that looking through the Sights you can ſee the top of the Tower exactly; then will the Label cut in the Limb of the Quadrant the Degrees of the Angle B A D, for the Limb is ſuppos'd to be graduated for this purpoſe: Then change your Station, moving in a Right-line forwards 10 Fathom (or it might have been any other Diſtance, and backward as well as forward) to C, and there take after the ſame manner the An-



gle B C D: By which means you will have alſo the Angle B C A, becauſe thoſe two together make 180 *deg.* or two Right ones. So that in the Triangle A C B you have now found the Baſe A C which is 10 Fathom, and alſo the two Angles at the Baſe; and conſequently the ſides C B and A B may be known (9. 18). And then you may have the Height D B or the diſtance A D, if you make a little Triangle Similar to that, and there from the Point *b* let fall a Perpendicular *b d* to the Baſe Line A C continued to *d.* For B D or A D will be juſt as many Fathoms as *b d* and *a d* will be equal Parts meaſur'd on Scale, (as in the laſt.) And if after you have thus gain'd the Height B D, you find, by the ſame method, the Height E D alſo, you may (*by Subſtracting this Altitude from the Former*) find the Height of the Tower E B.

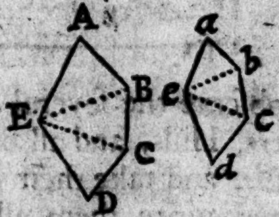
> N. B. (*The common Quadrant with a String and Plummet, and with the Sights fixt on one of its ſides, is more convenient and ready than this of* Pardies, *which is now out of Uſe*).

" Some times inſtead of advancing towards the
" Tower, and of making Obſervations of the Height
" below; or of thoſe Angles the viſual Rays make
" with the Horizontal Line, it is convenient to take
" two Stations ſide ways of each other; But it comes
" all to the ſame, and the Practices in reality are not
" at all different.

" And by this means, as any one may ſee, may
" all imaginable Heights and Diſtances and other
" Dimenſions be taken; provided we can but come
" to obſerve their Extremities, from two different
" Places. I ſhall not ſtay now to deſcribe the Par-
" ticular ways of doing this; nor to enumerate the
" great Advantages that would accrue from the uſe

“ of Teleſcopical Sights fixt on the Label, or on the “ ſide of the Inſtrument uſed in taking Angles; “ which indeed is an Invention of ineſtimable Benefit “ to Surveyors.

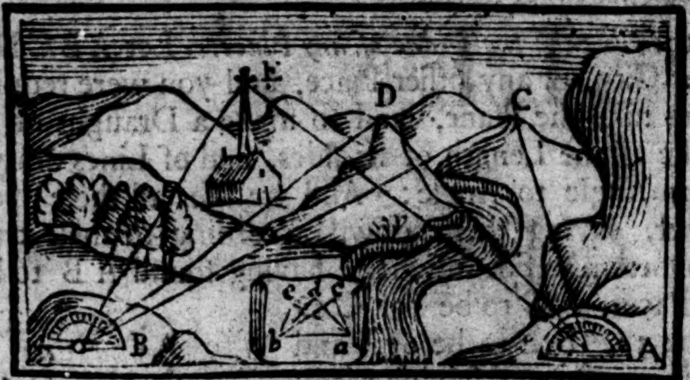
10. *To take the Plane of any Place.* Let ABCDE be a City, or any other Place, and you were required to take the Plane, and to make a Draught of it. Take all the Length of its ſides, and of Lines drawn from Angle to Angle: And transfer all theſe upon Paper, laying them down according to their true Proportion For Inſtance, having found A B to be 30 Paces, B C to be 59, C D to be 50, B E to be 67, and A E 49, *&c.* and having ready drawn on Paper a plain Scale divided into 100 equal Parts. Make the Line *a b* 30 of ſuch Parts; *b e*, 67; and *a.e*, 49, then thoſe Lines drawn and join'd together will make the Triangle *a b e* every way Similar to the Triangle A B E. And if you go on thus and make the Triangle *b e c* Similar to B E C, *&c.* you will form the Figure *a b c d e* every way Similar to the Plane of the Place A B C D E.



11. But if you cannot get into the Place to Survey it, and to Meaſure the diſtance between the Angles E B and E C, you muſt take the ſeveral Angles of the Plane, and transfer them into your Draught; ſo that if the Angle B A E be 66 *deg.* the Angle *b a e* muſt alſo be 66 *deg.* and ſo of all the reſt.

12. *To make a draught of any City or Country.* Aſcend up into any two elevated Places from whence you can plainly ſee the City or Country whoſe Delineation you would make. And having with you a Quadrant, whole Circle, or Semicircle well divided into Degrees, together with its Label (with Sights) and its Center: Place your Inſtrument at A, and

ſo that one of its ſides may lie in a Line between A and B, which done, and the Inſtrument fixt there,



obſerve the ſeveral Steeples, eminent Houſes, Towers, Hills, and all other remarkable Places as E D C, *&c.* and take their Angles with the Label and Sights, and write them all down to help your Memory. Thus, let the Angle C A B be 50 *deg.* 30 *min.* the Angle D A B 45 *dig.* 8 *min.* &c. Proceed after the ſame manner at the Station B; noting down the Angle A B C to be 40 *deg.* 10 *min.* the Angle A B D 47 *deg.* 28 *min.* &c. After which draw on Paper any Line at Pleaſure as *a b* and make at each end of it Angles equal to thoſe which you found *c a b* equal to C A B, *d a b* equal to D A B, and *a b c* equal to A B C, &c. and by this means you will have the Points *c*, *d* and *e*, &c which will be in the ſame Poſition to one another as the Steeples, or other eminent Places C D E, *&c.* are. And thus having drawn the moſt Conſpicuous and Principal Places the reſt may eaſily be taken by the Eye. But to make this Operation very exact, 'tis convenient to take the Angles alſo at a third or fourth Station, and then if they all agree, any one will know that the Work was well done.

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